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Clément Erignoux, Marielle Simon

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Abstract

We investigate the macroscopic energy diffusion of a disordered harmonic chain of oscillators, whose hamiltonian dynamics is perturbed by a degenerate conservative noise. After rescaling space and time diffusively, we prove that the equilibrium energy fluctuations evolve according to a linear heat equation. The diffusion coefficient is obtained from Varadhan's non-gradient approach, and is equivalently defined through the Green-Kubo formula. Since the perturbation is very degenerate and the symmetric part of the generator does not have a spectral gap, the standard non-gradient method is reviewed under new perspectives.

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1 Introduction

This paper deals with diffusive behaviour in heterogeneous media for interacting particle systems. More precisely, we address the problem of energy fluctuations for chains of oscillators with random defects. In the last fifty years, it has been recognized that introducing disorder in interacting particle systems has a drastic effect on the conduction properties of the material [8]. The most mathematically tractable model of oscillators is the one-dimensional system with harmonic interactions [1]. The anharmonic case is poorly understood from a mathematical point of view, but since the works of Peierls [24, 25], it is

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admitted that non-linear interactions between atoms should play a crucial role in the derivation of the Fourier law. In [2, 5, 7] (among many others) the authors propose to model anharmonicity by stochastic perturbations, in order to recover the expected macroscopic behavior: in some sense, the noise simulates the effect of non-linearities. Being inspired by all these previous works, the aim of this paper is to prove the diffusive energy behavior of disordered harmonic chains perturbed by an energy conserving noise. Moreover we prove that all the disorder effects are, on a sufficiently large scale, contained in a diffusion coefficient, which depends on the statistics of the field, but not on the randomness itself.

On the one hand, the disorder effect has already been investigated for *lattice gas dynamics*: the first article dealing with scaling limits of particle systems in random environment is the remarkable work of Fritz [12], and since then the subject has attracted a lot of interest, see for example [11, 15, 22, 26]. These papers share one main feature: the models are *non-gradient*[1] due to the presence of the disorder. Except in [12], non-gradient systems are usually solved by establishing a microscopic Fourier law up to a small fluctuating term, following the sophisticated method initially developed by Varadhan in [30], and generalized to non-reversible dynamics in [17]. These previous works mostly deal with systems of particles that evolve according to an exclusion process in random environment: the particles are attempting jumps to nearest neighbour sites at rates which depend on both their position and the objective site, and the rates themselves come from a quenched random field. Different approaches are adopted to tackle the non-gradient feature: whereas the standard method of Varadhan is helpful in dimension $d \geqslant 3$ only (see [11]), the "long jump" variation developed by Quastel in [26] is valid in any dimension. The study of disordered chains of oscillators perturbed by a conservative noise has appeared more recently, see for instance [3, 4, 9]. In all these papers, the thermal conductivity is defined by the Green-Kubo formula only. Here, we also define the diffusion coefficient through hydrodynamics and we prove that both definitions are equivalent.

On the other hand, the study of *one-dimensional chains of oscillators* is an active field of research. In [19], the authors derive the diffusive scaling limit for a *homogeneous* (without disorder) chain of coupled harmonic oscillators, perturbed by a noise which randomly flips the sign of the velocities (called *velocity-flip noise*), so that the energy is conserved but not the momentum. We want to investigate here the scaling limit of equilibrium fluctuations for the same chain of harmonic oscillators, still perturbed by the velocity-flip noise, but now provided with i.i.d. random masses. In [29], for the same model, an exact *fluctuation-dissipation relation* (see for example [20]) reproduces the Fourier law at the microscopic level. With random masses, however, the fluctuation-dissipation equations are no longer directly solvable. We therefore adapt Varadhan's non-gradient approach, which allows one to show that an approximate fluctuation-dissipation decomposition holds. The main ingredients of the usual non-reversible non-gradient method are: first, a *spectral gap estimate* for the symmetric part of the dynamics, and second, a *sector condition* for the total generator. The rigorous study of the disordered harmonic chain perturbed by the velocity-flip noise contains three major obstacles: (i) first, the symmetric part of the generator (which, in our case, comes only from the stochastic noise) is poorly ergodic, and does not have a spectral gap when restricted to micro-canonical manifolds. This issue is usually critical to apply Varadhan's method ; (ii) second, the degeneracy of the perturbation implies that the asymmetric part of the generator cannot be controlled by its symmetric part (in technical terms, the sector condition does not hold) ; (iii) finally, the energy current depends on the disorder, and has to be approximated by a fluctuation-dissipation equation which takes into account the fluctuations of the disorder itself.

[1] Roughly speaking, the *gradient property* states that the microscopic current (of density, or energy, depending on the conservation law under consideration) can be decomposed as a local gradient. We refer to Section 2.2 for more details.

To overcome the second obstacle (ii), namely the lack of sector condition due to the high degeneracy of the velocity-flip noise, we add a second stochastic perturbation, that exchanges velocities (divided by the square root of mass) and positions at random independent Poissonian times, so that a kind of sector condition can be proved (see Proposition 5.7: we call it the *weak sector condition*). However, the spectral gap estimate and the usual sector condition still do not hold when adding the exchange noise, meaning that the stochastic perturbation remains very degenerate; in other words, the noises are still far from ergodic. To sum up, the final model that we rigorously investigate here is: the coupled harmonic chain with random masses, perturbed by two degenerate stochastic noises, one which exchanges velocities and positions, the other one which flips the sign of velocities. The main results and contributions of this article are

- an adaptation of the non-gradient method to a microscopic model for which neither the spectral gap inequality nor the sector condition hold (Theorem 3.3 and Theorem 5.9), which makes no use of the closed forms theory. In particular, the main novelty is Lemma 4.3 ;
- the macroscopic behavior of the equilibrium fluctuations of the energy, which is diffusive, with a diffusion coefficient D depending only on the statistics of the random masses field (Theorem 3.1) ;
- the equivalence between two definitions of the diffusion coefficient: the one obtained via Varadhan's approach, and the one obtained from the Green-Kubo formula (Theorem 7.3).

Our model has one crucial feature, that makes an adaptation of Varadhan's approach possible despite the lack of spectral gap of the symmetric part of the generator: thanks to the harmonicity of the chain, the generator of the dynamics preserves homogeneous polymonials together with their degree. In particular, the derivation of the sector condition and the non-gradient decomposition of closed discrete differential forms at the center of the non-gradient method (see [28, 16] for more details) can be carried out rather explicitly in a suitable space of quadratic functions, without need for a spectral gap. Although some further complications appear in the study of our model, this is one of its clear advantages. It allows us to avoid significant technical difficulties usually inherent to Varadhan's approach, and to adapt the latter to a model with a very degenerate noise. In particular, if the chain is not assumed to be harmonic, a stronger noise than ours is generally needed: the one proposed by Olla and Sasada in [23] is strong enough so that the spectral gap and the sector condition hold, and they were able to use ideas from Varadhan's approach to determine the scaling limit of the equilibrium fluctuations. Our purpose here is to show, using elements of the non-gradient method as well, that in the presence of i.i.d. random masses, the annealed (i.e. averaged out over the masses' randomness) equilibrium fluctuations of the energy evolve following an infinite Ornstein-Uhlenbeck process. The covariances characterizing this linearised heat equation are given in terms of the diffusion coefficient, which is defined through a variational formula. We opted for a rather detailed redaction, even if some proofs may look standard to expert readers. We hope that this choice will be beneficial for the reader not already familiar with the non-gradient method.

Finally, we also show that the diffusion coefficient can be equivalently given by the Green-Kubo formula. The latter is defined as the space-time variance of the current at equilibrium, which is only formal in the sense that a double limit (in space and time) has to be taken. As in [3], where the disordered harmonic chain is perturbed by a stronger energy conserving noise, we prove here that the limit exists, and that the homogenization effect occurs for the Green-Kubo formula: for almost every realization of the disorder, the thermal conductivity exists, is independent of the disorder, is positive and finite. This allows us to prove that the diffusion coefficient D obtained through the variational formula in Varadhan's method, and the coefficient $\overline{\mathrm{D}}$ defined through the Green-Kubo formula, are actually equal: $\mathrm{D} = \overline{\mathrm{D}}$.

To conclude this introduction, we introduce in more details the model on which this article focuses.

As explained earlier, we consider here an infinite harmonic hamiltonian system described by the sequence $\{p_x, r_x\}_{x\in\mathbb{Z}}$, where p_x stands for the momentum of the oscillator at site x, and r_x represents the distance between oscillator x and oscillator $x+1$. Each atom $x \in \mathbb{Z}$ has a mass $\mathrm{M}_x > 0$, thus the velocity of atom x is given by p_x/M_x. We assume the disorder $\mathbf{M} := \{\mathrm{M}_x\}_{x\in\mathbb{Z}}$ to be a collection of real i.i.d. positive random variables such that

$$\forall\, x \in \mathbb{Z}, \quad \frac{1}{\mathrm{C}} \leqslant \mathrm{M}_x \leqslant \mathrm{C}, \tag{1}$$

for some finite constant $\mathrm{C} > 0$. The equations of motions are given by

$$\begin{cases} \dfrac{\mathrm{d}p_x}{\mathrm{d}t} = r_x - r_{x-1}, \\ \dfrac{\mathrm{d}r_x}{\mathrm{d}t} = \dfrac{p_{x+1}}{\mathrm{M}_{x+1}} - \dfrac{p_x}{\mathrm{M}_x}, \end{cases} \tag{2}$$

so that the dynamics conserves the total energy

$$\mathcal{E} := \sum_{x\in\mathbb{Z}} \left\{ \frac{p_x^2}{2\mathrm{M}_x} + \frac{r_x^2}{2} \right\}.$$

To overcome the lack of ergodicity of deterministic chains[2], we add a stochastic perturbation to (2). The noise can be easily described: at independently distributed random Poissonian times, the quantity $p_x/\sqrt{\mathrm{M}_x}$ and the interdistance r_x are exchanged, or the momentum p_x is flipped into $-p_x$. This noise still conserves the total energy $\mathcal{E}$, and is very degenerate. The main goal of this paper is to prove that the energy fluctuations in equilibrium converge in a suitable space-time scaling limit (Theorem 3.1).

Even if Theorem 3.1 could be proved *mutatis mutandis* for this harmonic chain described by $\{p_x, r_x\}$, for pedagogical reasons we now focus on a simplified model, which has the same features and involves simplified computations[3]. From now on, we study the dynamics on new configurations $\{\eta_x\}_{x\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ written as

$$m_x \mathrm{d}\eta_x = (\eta_{x+1} - \eta_{x-1})\mathrm{d}t, \tag{3}$$

where $\mathbf{m} := \{m_x\}_{x\in\mathbb{Z}}$ is the new disorder with the same characteristics as in (1). It is notationally convenient to change the variable η_x into $\omega_x := \sqrt{m_x}\eta_x$, so that the total energy reads

$$\mathcal{E} = \sum_{x\in\mathbb{Z}} \omega_x^2.$$

Let us now introduce the corresponding stochastic energy conserving dynamics: the evolution is described by (3) between random exponential times, and at each ring one of the following interactions can happen:

a. Exchange noise – the two nearest neighbour variables ω_x and ω_{x+1} are exchanged;

b. Flip noise – the variable ω_x at site x is flipped into $-\omega_x$.

As a consequence of these two perturbations, the dynamics only conserves the total energy, the other important conservation laws of the hamiltonian part being destroyed by the stochastic noises[4]. It is not

[2] For the deterministic system of harmonic oscillators, it is well known that the energy is ballistic, destroying the validity of the Fourier law. For more details, see the remarkable work of Lebowitz, Lieb and Rieder [21], which is the standard reference.

[3] We invite the reader to see [6] for the origin of this new particle system.

[4] It is now well understood that the ballisticity of the harmonic chain is due to the infinite number of conserved quantities. In 1994, Fritz, Funaki and Lebowitz [13] propose different stochastic noises that are sufficient to destroy the ballisticity of the chain: the *velocity-flip* noise is one of them.

difficult to check that the following family $\{\mu_\beta\}_{\beta>0}$ of *grand-canonical Gibbs measures* on $\mathbb{R}^{\mathbb{Z}}$ is invariant for the resulting process $\{\omega_x(t)\ ;\ x\in\mathbb{Z}, t\geqslant 0\}$:

$$\mu_\beta(\mathrm{d}\omega) := \prod_{x\in\mathbb{Z}} \sqrt{\frac{\beta}{2\pi}} \exp\left(-\frac{\beta}{2}\omega_x^2\right) \mathrm{d}\omega_x. \tag{4}$$

The index β stands for the inverse temperature. Note that with our notational convenience, μ_β does not depend on the disorder $\mathbf{m}$. Observe also that the dynamics is not reversible with respect to the measure μ_β. We define $\mathbf{e}_\beta := \beta^{-1}$ as the thermodynamical energy associated to β, namely the expectation of ω_0^2 with respect to μ_β, and $\chi(\beta) = 2\beta^{-2}$ as the static compressibility, namely the variance of ω_0^2 with respect to μ_β.

To state the convergence result, let us define the distribution-valued energy fluctuation field, as follows: at time 0, it is given by

$$\mathcal{Y}_0^{\mathrm{N}} := \frac{1}{\sqrt{\mathrm{N}}} \sum_{x\in\mathbb{Z}} \delta_{x/\mathrm{N}} \{\omega_x^2(0) - \mathbf{e}_\beta\},$$

where δ_u is the Dirac measure at point $u\in\mathbb{R}$. We assume that the dynamics is at equilibrium, namely that $\{\omega_x(0)\}_{x\in\mathbb{Z}}$ is distributed according to the Gibbs measure μ_β. It is well known that $\mathcal{Y}_0^{\mathrm{N}}$ converges in distribution as $\mathrm{N}\to\infty$ towards a centered Gaussian field $\mathcal{Y}$, which satisfies

$$\mathbb{E}_{\mathcal{Y}}\big[\mathcal{Y}(\mathrm{F})\mathcal{Y}(\mathrm{G})\big] = \chi(\beta) \int_{\mathbb{R}} \mathrm{F}(y)\mathrm{G}(y)\mathrm{d}y,$$

for continuous test functions F, G. One of the main results of this article, Theorem 3.1 below, states that the energy fluctuations evolve diffusively in time: starting from μ_β, and averaging over the disorder $\mathbf{m}$, the energy field

$$\mathcal{Y}_t^{\mathrm{N}} = \frac{1}{\sqrt{\mathrm{N}}} \sum_{x\in\mathbb{Z}} \delta_{x/\mathrm{N}} \{\omega_x^2(t\mathrm{N}^2) - \mathbf{e}_\beta\}$$

converges in distribution as $\mathrm{N}\to\infty$ to the solution of the linear Stochastic Partial Differential Equation (SPDE)

$$\partial_t\mathcal{Y} = \mathrm{D}\partial_y^2\mathcal{Y} + \sqrt{2\mathrm{D}\chi(\beta)}\partial_y \mathrm{W}, \qquad t>0, y\in\mathbb{R}.$$

where D is the diffusion coefficient, defined by variational formula (see Definition 2.4 below), and W is the standard normalized space-time white noise.

We note that one could think of using the well-known *entropy method* [14] to further derive the *hydrodynamic equation*: in that case, the initial law is not assumed to be the equilibrium measure μ_β, but a *local equilibrium measure* (see (81) below). We conjecture that this property of local equilibrium propagates in time, and that an *hydrodynamic limit* result holds. In other words, let $\mathbf{e}_0 : \mathbb{T}\to\mathbb{R}$ be a bounded function, where $\mathbb{T}$ denotes the torus $[0,1)$. The problem is to show that the empirical energy profile $\frac{1}{\mathrm{N}}\sum_x \delta_{x/\mathrm{N}}\, \omega_x^2(t\mathrm{N}^2)$ converges as $\mathrm{N}\to\infty$ to the macroscopic profile $\mathbf{e}(t,\cdot):\mathbb{T}\to\mathbb{R}$ solution to

$$\begin{cases} \dfrac{\partial \mathbf{e}}{\partial t}(t,u) = \mathrm{D}\dfrac{\partial^2 \mathbf{e}}{\partial u^2}(t,u), & t>0,\ u\in\mathbb{T}, \\ \mathbf{e}(0,u) = \mathbf{e}_0(u). \end{cases}$$

Unfortunately, even if the diffusion coefficient D is well defined through the non-gradient approach, this does not straightforwardly provide a method to prove such a result. This topic is discussed in Section 9.

Let us now give the plan of the article. Section 2 is devoted to properly introducing the model and all definitions that are needed. The convergence of the energy fluctuations field (in the sense of finite dimensional distributions) is stated and proved in Section 3. The main point is to identify the diffusion coefficient D (Section 5), by adapting the non-gradient method of Varadhan [30]. This is done in several steps: in Section 4, we derive the so-called *Boltzmann-Gibbs principle*, in Section 5 we obtain the diffusion coefficient as resulting from a projection of the current in some suitable Hilbert space, and finally Section 6 improves the description of the diffusion coefficient through several variational formulas. In Section 7 we prove the convergence of the Green-Kubo formula, and demonstrate rigorously that both definitions of the diffusion coefficient are equivalent. In Section 8, we present a second disordered model, where the interaction is described by a potential V which is not assumed to be harmonic anymore. For this anharmonic chain, we need a very strong stochastic perturbation, which has a spectral gap, and satisfies the sector condition. We conclude in Section 9 by highlighting the step where the usual techniques for proving hydrodynamic limits fail. In Appendices, technical points are detailed: in Appendix A, the space of square-integrable functions w.r.t. the standard Gaussian law is studied through its orthonormal basis of Hermite polynomials. The sector condition is proved in Appendix B for a specific class of functions suitable for our needs. In Appendix C, the tightness for the energy fluctuation field is investigated for the sake of completeness.

2 The harmonic chain perturbed by stochastic noises

2.1 Generator of the Markov process

Let us define the dynamics on the finite torus $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$, meaning that boundary conditions are periodic. The space of configurations is given by $\Omega_N = \mathbb{R}^{\mathbb{T}_N}$. The configuration $\{\omega_x\}_{x\in\mathbb{T}_N}$ evolves according to a dynamics which can be divided into two parts, a deterministic one and a stochastic one. The disorder is an i.i.d. sequence $\mathbf{m} = \{m_x\}_{x\in\mathbb{Z}}$ which satisfies:

$$\forall\, x \in \mathbb{Z}, \quad \frac{1}{C} \leqslant m_x \leqslant C,$$

for some finite constant $C > 1$. The corresponding product and translation invariant measure on the space $\Omega_{\mathcal{D}} = [C^{-1}, C]^{\mathbb{Z}}$ is denoted by $\mathbb{P}$ and its expectation is denoted by $\mathbb{E}$. For a fixed disorder field $\mathbf{m} = \{m_x\}_{x\in\mathbb{Z}}$, we consider the system

$$\sqrt{m_x}\,\mathrm{d}\omega_x = \left(\frac{\omega_{x+1}}{\sqrt{m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1}}}\right)\mathrm{d}t, \qquad t \geqslant 0,\ x \in \mathbb{T}_N,$$

and we superpose to this deterministic dynamics a stochastic perturbation described as follows: with each atom $x \in \mathbb{T}_N$ (respectively each bond $\{x, x+1\}, x \in \mathbb{T}_N$) is associated an exponential clock of rate $\gamma > 0$ (resp. $\lambda > 0$), and all clocks are independent one from another. When the clock attached to the atom x rings, ω_x is flipped into $-\omega_x$. When the clock attached to the bond $\{x, x+1\}$ rings, the values ω_x and ω_{x+1} are exchanged. This dynamics can be equivalently defined by the generator $\mathcal{L}_N^{\mathbf{m}}$ of the Markov process $\{\omega_x(t)\,;\, x \in \mathbb{T}_N\}_{t\geqslant 0}$, which is written as

$$\mathcal{L}_N^{\mathbf{m}} = \mathcal{A}_N^{\mathbf{m}} + \gamma \mathcal{S}_N^{\text{flip}} + \lambda \mathcal{S}_N^{\text{exch}},$$

where

$$\mathcal{A}_N^{\mathbf{m}} = \sum_{x\in\mathbb{T}_N} \left(\frac{\omega_{x+1}}{\sqrt{m_x m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1} m_x}}\right)\frac{\partial}{\partial\omega_x},$$

and, for all functions $f : \Omega_{\mathcal{D}} \times \Omega_{\mathrm{N}} \to \mathbb{R}$,

$$\mathcal{S}_{\mathrm{N}}^{\mathrm{flip}} f(\mathbf{m},\omega) = \sum_{x\in\mathbb{T}_{\mathrm{N}}} \Big(f(\mathbf{m},\omega^x) - f(\mathbf{m},\omega)\Big),$$
$$\mathcal{S}_{\mathrm{N}}^{\mathrm{exch}} f(\mathbf{m},\omega) = \sum_{x\in\mathbb{T}_{\mathrm{N}}} \Big(f(\mathbf{m},\omega^{x,x+1}) - f(\mathbf{m},\omega)\Big).$$

Here, the configuration ω^x is the configuration obtained from ω by flipping the value at site x:

$$(\omega^x)_z = \begin{cases} \omega_z & \text{if } z \neq x, \\ -\omega_x & \text{if } z = x, \end{cases}$$

and the configuration $\omega^{x,x+1}$ is obtained from ω by exchanging the values at sites x and $x+1$:

$$(\omega^{x,x+1})_z = \begin{cases} \omega_z & \text{if } z \neq x, x+1, \\ \omega_{x+1} & \text{if } z = x, \\ \omega_x & \text{if } z = x+1. \end{cases}$$

We denote the total generator of the noise by $\mathcal{S}_{\mathrm{N}} := \gamma\mathcal{S}_{\mathrm{N}}^{\mathrm{flip}} + \lambda\mathcal{S}_{\mathrm{N}}^{\mathrm{exch}}$.

It is straightforward to see that the total energy $\sum \omega_x^2$ is conserved by the dynamics and that the following translation invariant product Gibbs measures μ_β^{N} on Ω_{N} are invariant for the process:

$$\mathrm{d}\mu_\beta^{\mathrm{N}}(\omega) := \prod_{x\in\mathbb{T}_{\mathrm{N}}} \sqrt{\frac{\beta}{2\pi}} \exp\left(-\frac{\beta}{2}\omega_x^2\right) \mathrm{d}\omega_x.$$

The index β stands for the inverse temperature, namely $\int \omega_0^2 \mathrm{d}\mu_\beta^{\mathrm{N}} = \beta^{-1}$. Let us note that the Gibbs measures do not depend on the disorder $\mathbf{m}$. From the definition, our model is not reversible with respect to the measure μ_β^{N}. More precisely, $\mathcal{A}_{\mathrm{N}}^{\mathbf{m}}$ is an antisymmetric operator in $\mathbf{L}^2(\mu_\beta^{\mathrm{N}})$, whereas $\mathcal{S}_{\mathrm{N}}$ is symmetric.

NOTATIONS – In the following, we denote by Ω the space of configurations in the infinite line, that is $\Omega := \mathbb{R}^{\mathbb{Z}}$, and by μ_β the infinite product Gibbs measure on $\mathbb{R}^{\mathbb{Z}}$. The natural scalar product in $\mathbf{L}^2(\mu_\beta)$ is denoted by $\langle \cdot,\cdot\rangle_\beta$. Moreover, we denote by $\mathbb{P}_\beta^\star$ the probability measure on $\Omega_{\mathcal{D}} \times \Omega$ defined by

$$\mathbb{P}_\beta^\star := \mathbb{P} \otimes \mu_\beta.$$

Throughout this article we will widely use the fact that $\mathbb{P}_\beta^\star$ is translation invariant. We write $\mathbb{E}_\beta^\star$ for the corresponding expectation, and $\mathbb{E}_\beta^\star[\cdot,\cdot]$ for the scalar product in $\mathbf{L}^2(\mathbb{P}_\beta^\star)$. We also define the static compressibility which is equal to the variance of the one-site energy ω_0^2 with respect to μ_β, namely

$$\chi(\beta) := \langle \omega_0^4\rangle_\beta - \langle\omega_0^2\rangle_\beta^2 = \frac{2}{\beta^2}.$$

2.2 Energy current

Since the dynamics conserves the total energy, there exist instantaneous currents of energy $j_{x,x+1}$ such that $\mathcal{L}_{\mathrm{N}}^{\mathbf{m}}(\omega_x^2) = j_{x-1,x}(\mathbf{m},\omega) - j_{x,x+1}(\mathbf{m},\omega)$. The quantity $j_{x,x+1}$ is the instantaneous amount of energy flowing between the particles x and $x+1$, and is equal to

$$j_{x,x+1}(\mathbf{m},\omega) = -\frac{2\omega_x\omega_{x+1}}{\sqrt{m_x m_{x+1}}} + \lambda(\omega_x^2 - \omega_{x+1}^2).$$

We write $j_{x,x+1} = j^{\mathrm{A}}_{x,x+1} + j^{\mathrm{S}}_{x,x+1}$ where $j^{\mathrm{A}}_{x,x+1}$ (resp. $j^{\mathrm{S}}_{x,x+1}$) is the current associated to the antisymmetric (resp. symmetric) part of the generator:

$$\begin{aligned} j^{\mathrm{A}}_{x,x+1}(\mathbf{m},\omega) &= -\frac{2\omega_x\omega_{x+1}}{\sqrt{m_x m_{x+1}}} \\ j^{\mathrm{S}}_{x,x+1}(\mathbf{m},\omega) &= j^{\mathrm{S}}_{x,x+1}(\omega) = \lambda(\omega_x^2 - \omega_{x+1}^2). \end{aligned}$$

As mentionned in the introduction, this model is *non-gradient*, i.e. the current cannot be directly written as the gradient of a local function. Moreover, there is not an exact *fluctuation-dissipation equation*, as in [29].

2.3 Cylinder functions

For every $x \in \mathbb{Z}$ and every measurable function f on $\Omega_{\mathcal{D}} \times \Omega$, we define the translated function $\tau_x f$ on $\Omega_{\mathcal{D}} \times \Omega$ by: $\tau_x f(\mathbf{m},\omega) := f(\tau_x\mathbf{m}, \tau_x\omega)$, where $\tau_x\mathbf{m}$ and $\tau_x\omega$ are the disorder and particle configurations translated by $x \in \mathbb{Z}$, respectively:

$$(\tau_x\mathbf{m})_z := m_{x+z}, \qquad (\tau_x\omega)_z = \omega_{x+z}.$$

Let Λ be a finite subset of $\mathbb{Z}$, and denote by $\mathcal{F}_\Lambda$ the σ-algebra generated by $\{m_x, \omega_x \ ; \ x \in \Lambda\}$. For a fixed positive integer ℓ, we define $\Lambda_\ell := \{-\ell, \dots, \ell\}$. If the box is centered at site $x \neq 0$, we denote it by $\Lambda_\ell(x) := \{-\ell + x, \dots, \ell + x\}$. If f is a measurable function on $\Omega_{\mathcal{D}} \times \Omega$, the support of f, denoted by Λ_f, is the smallest subset of $\mathbb{Z}$ such that $f(\mathbf{m},\omega)$ only depends on $\{m_x, \omega_x \ ; \ x \in \Lambda_f\}$ and f is called a cylinder (or local) function if Λ_f is finite. In that case, we denote by s_f the smallest positive integer s such that Λ_s contains the support of f and then $\Lambda_f = \Lambda_{s_f}$. For every cylinder function $f : \Omega_{\mathcal{D}} \times \Omega \to \mathbb{R}$, consider the formal sum

$$\Gamma_f := \sum_{x \in \mathbb{Z}} \tau_x f$$

which is ill defined, but for which both gradients

$$\begin{aligned} \nabla_0(\Gamma_f) &:= \Gamma_f(\mathbf{m},\omega^0) - \Gamma_f(\mathbf{m},\omega), \\ \nabla_{0,1}(\Gamma_f) &:= \Gamma_f(\mathbf{m},\omega^{0,1}) - \Gamma_f(\mathbf{m},\omega), \end{aligned}$$

only involve a finite number of non-zero contributions and are therefore well defined. Similarly, we define for any $x \in \mathbb{T}_N$

$$\begin{aligned} (\nabla_x f)(\mathbf{m},\omega) &:= f(\mathbf{m},\omega^x) - f(\mathbf{m},\omega), \\ (\nabla_{x,x+1} f)(\mathbf{m},\omega) &:= f(\mathbf{m},\omega^{x,x+1}) - f(\mathbf{m},\omega). \end{aligned}$$

Definition 2.1. *We denote by $\mathcal{C}$ the set of cylinder functions φ on $\Omega_{\mathcal{D}} \times \Omega$, such that*

(i) for all $\omega \in \Omega$, the random variable $\mathbf{m} \mapsto \varphi(\mathbf{m},\omega)$ is continuous on $\Omega_{\mathcal{D}}$;

(ii) for all $\mathbf{m} \in \Omega_{\mathcal{D}}$, the function $\omega \mapsto \varphi(\mathbf{m},\omega)$ belongs to $\mathbf{L}^2(\mu_\beta)$ and has mean zero with respect to μ_β.

Definition 2.2. *We introduce the set of quadratic cylinder functions on $\Omega_{\mathcal{D}} \times \Omega$, denoted by $\mathcal{Q} \subset \mathcal{C}$, and defined as follows: $f \in \mathcal{Q}$ if there exists a sequence $\{\psi_{i,j}(\mathbf{m})\}_{i,j \in \mathbb{Z}}$ of cylinder functions on $\Omega_{\mathcal{D}}$, with finite support in $\Omega_{\mathcal{D}}$, such that*

(i) for all $i, j \in \mathbb{Z}$ and all $\omega \in \Omega$, the random variable $\mathbf{m} \mapsto \psi_{i,j}(\mathbf{m})$ is continuous on $\Omega_{\mathcal{D}}$;

(ii) $\psi_{i,j}$ *vanishes for all but a finite number of pairs* (i,j)*, and is symmetric:* $\psi_{i,j} = \psi_{j,i}$ *;*

(iii) f *is written as*

$$f(\mathbf{m},\omega) = \sum_{i\in\mathbb{Z}} \psi_{i,i}(\mathbf{m})(\omega_{i+1}^2 - \omega_i^2) + \sum_{\substack{i,j\in\mathbb{Z}\\ i\neq j}} \psi_{i,j}(\mathbf{m})\omega_i\omega_j. \tag{5}$$

One easily checks that $\mathcal{Q}$ is invariant under the action of the generator $\mathcal{L}_{\mathrm{N}}^{\mathbf{m}}$. In other words, quadratic functions are homogeneous polynomials of degree two in the variable ω, that have mean zero with respect to μ_β for every $\mathbf{m} \in \Omega_{\mathcal{D}}$. Another definition through *Hermite polynomials* is given in Appendix A (see Section A.3). We are now ready to define two sets of functions that will play a crucial role later on.

Definition 2.3. *Let* $\mathcal{C}_0$ *be the set of cylinder functions* φ *on* $\Omega_{\mathcal{D}} \times \Omega$ *such that there exists a finite subset* Λ *of* $\mathbb{Z}$*, and cylinder, measurable functions* $\{\mathrm{F}_x, \mathrm{G}_x\}_{x\in\Lambda}$ *defined on* $\Omega_{\mathcal{D}} \times \Omega$*, that verify*

$$\varphi = \sum_{x\in\Lambda} \Big\{ \nabla_x(\mathrm{F}_x) + \nabla_{x,x+1}(\mathrm{G}_x) \Big\},$$

and such that, for all $x \in \Lambda$*,*

(i) for all $\omega \in \Omega$*, the functions* $\mathbf{m} \mapsto \mathrm{F}_x(\mathbf{m},\omega)$ *and* $\mathbf{m} \mapsto \mathrm{G}_x(\mathbf{m},\omega)$ *are continuous on* $\Omega_{\mathcal{D}}$*;*

(ii) for all $\mathbf{m} \in \Omega_{\mathcal{D}}$*, the functions* $\omega \mapsto \mathrm{F}_x(\mathbf{m},\omega)$ *and* $\omega \mapsto \mathrm{G}_x(\mathbf{m},\omega)$ *belong to* $\mathbf{L}^2(\mu_\beta)$*.*

Let $\mathcal{Q}_0 \subset \mathcal{C}_0 \cap \mathcal{Q}$ *be the set of such functions* φ*, with the additional assumption that the cylinder functions* F_x*,* G_x *are homogeneous polynomials of degree two in the variable* ω *(but not necessarily with mean zero as before).*

Finally, we introduce the infinite volume counterparts of $\mathcal{L}_{\mathrm{N}}^{\mathbf{m}}$, $\mathcal{A}_{\mathrm{N}}^{\mathbf{m}}$, $\mathcal{S}_{\mathrm{N}}^{\mathrm{flip}}$ and $\mathcal{S}_{\mathrm{N}}^{\mathrm{exch}}$, namely the operators $\mathcal{L}^{\mathbf{m}}$, $\mathcal{A}^{\mathbf{m}}$, $\mathcal{S}^{\mathrm{flip}}$ and $\mathcal{S}^{\mathrm{exch}}$, acting on cylinder functions f on $\Omega_{\mathcal{D}} \times \Omega$:

$$\mathcal{L}^{\mathbf{m}} f = \mathcal{A}^{\mathbf{m}} f + \gamma\mathcal{S}^{\mathrm{flip}} f + \lambda\mathcal{S}^{\mathrm{exch}} f$$

with

$$\mathcal{A}^{\mathbf{m}} = \sum_{x\in\mathbb{Z}} \left(\frac{\omega_{x+1}}{\sqrt{m_x m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1} m_x}} \right) \frac{\partial}{\partial\omega_x},$$

and

$$\mathcal{S}^{\mathrm{flip}} f(\mathbf{m},\omega) = \sum_{x\in\mathbb{Z}} (\nabla_x f)(\mathbf{m},\omega) = \sum_{x\in\mathbb{Z}} \Big(f(\mathbf{m},\omega^x) - f(\mathbf{m},\omega) \Big),$$
$$\mathcal{S}^{\mathrm{exch}} f(\mathbf{m},\omega) = \sum_{x\in\mathbb{Z}} (\nabla_{x,x+1} f)(\mathbf{m},\omega) = \sum_{x\in\mathbb{Z}} \Big(f(\mathbf{m},\omega^{x,x+1}) - f(\mathbf{m},\omega) \Big).$$

We shorten for convenience

$$\mathcal{S} = \gamma\mathcal{S}^{\mathrm{flip}} + \lambda\mathcal{S}^{\mathrm{exch}}$$

the Markov generator giving the symmetric part of $\mathcal{L}^{\mathbf{m}}$. Given $\Lambda_\ell = \{-\ell,\dots,\ell\}$, we define $\mathcal{L}_{\Lambda_\ell}^{\mathbf{m}}$, resp. $\mathcal{S}_{\Lambda_\ell}$, as the restriction of the generator $\mathcal{L}^{\mathbf{m}}$, resp. $\mathcal{S}$, to Λ_ℓ. For the jump dynamics $\mathcal{S}^{\mathrm{exch}}$, we consider in $\mathcal{S}_{\Lambda_\ell}$ only the jumps with both ends in Λ_ℓ, namely

$$\mathcal{S}_{\Lambda_\ell} f(\mathbf{m},\omega) = \gamma \sum_{x\in\Lambda_\ell} (\nabla_x f)(\mathbf{m},\omega) + \lambda \sum_{x\in\Lambda_\ell\backslash\{\ell\}} (\nabla_{x,x+1} f)(\mathbf{m},\omega).$$

2.4 Dirichlet form and properties of $\mathcal{C}_0$ and $\mathcal{Q}_0$

Before giving the main properties of the sets introduced above, we introduce the usual quadratic form associated to the generator: for any $x \in \mathbb{Z}$, any cylinder functions $f, g \in \mathcal{C}$, and any positive integer ℓ, let us define

$$\mathcal{D}_\ell(\mu_\beta; f) := \big\langle (-\mathcal{L}^{\mathbf{m}}_{\Lambda_\ell}) f, f \big\rangle_\beta = \big\langle (-\mathcal{S}_{\Lambda_\ell}) f, f \big\rangle_\beta. \tag{6}$$

Since

$$\big\langle \nabla_x f, g \big\rangle_\beta = -\frac{1}{2} \langle \nabla_x f, \nabla_x g \rangle_\beta \quad \text{and} \quad \big\langle \nabla_{x,x+1} f, g \big\rangle_\beta = -\frac{1}{2} \langle \nabla_{x,x+1} f, \nabla_{x,x+1} g \rangle_\beta,$$

equation (6) in particular rewrites

$$\mathcal{D}_\ell(\mu_\beta; f) = \frac{\gamma}{2} \sum_{x \in \Lambda_\ell} \big\langle (\nabla_x f)^2 \big\rangle_\beta + \frac{\lambda}{2} \sum_{x \in \Lambda_\ell \setminus \{\ell\}} \big\langle \big(\nabla_{x,x+1} f\big)^2 \big\rangle_\beta. \tag{7}$$

The symmetric form $\mathcal{D}_\ell$ is called the *Dirichlet form*, and is well defined on $\mathcal{C}$. It is a random variable with respect to the disorder $\mathbf{m}$. Note that, since the symmetric part of the generator does not depend on $\mathbf{m}$, we can write

$$\mathbb{E}\big[\mathcal{D}_\ell(\mu_\beta; f)\big] = \mathcal{D}_\ell(\mathbb{P}^\star_\beta; f).$$

PROPOSITION 2.1. *The following elements belong to $\mathcal{Q}_0$:*

(a) $j^{\mathrm{S}}_{0,1}$, $j^{\mathrm{A}}_{0,1}$.

(b) $\mathcal{L}^{\mathbf{m}} f$, $\mathcal{S} f$ *and* $\mathcal{A}^{\mathbf{m}} f$, *for all* $f \in \mathcal{Q}$.

Proof. The first statement (a) is directly obtained from the following identities: for $x \in \mathbb{Z}$, and $k \geqslant 1$,

$$\omega_{x+1}^2 - \omega_x^2 = \nabla_{x,x+1}\left(\omega_x^2\right) \tag{8}$$

$$\omega_x \omega_{x+k} = -\frac{1}{2} \nabla_x \big(\omega_x \omega_{x+1}\big) + \sum_{\ell=1}^{k-1} \nabla_{x+\ell,x+\ell+1} \big(\omega_x \omega_{x+\ell}\big). \tag{9}$$

Then, if $f \in \mathcal{Q}$ is of the form (5), it is easy to see that (8) and (9) are sufficient to prove (b). For instance,

$$\mathcal{L}^{\mathbf{m}}(\omega_x \omega_{x+1}) = \frac{\omega_x \omega_{x+2}}{\sqrt{m_{x+1} m_{x+2}}} - \frac{\omega_{x+1}\omega_{x-1}}{\sqrt{m_x m_{x-1}}} + \frac{\omega_{x+1}^2 - \omega_x^2}{\sqrt{m_x m_{x+1}}} \\ - 4\gamma \omega_x \omega_{x+1} + \lambda(\omega_{x+2} - \omega_{x+1})\omega_x + \lambda(\omega_{x-1} - \omega_x)\omega_{x+1}.$$

The integrability and regularity conditions are straightforward. □

PROPOSITION 2.2 (Dirichlet bound). *Let φ be a cylinder function in $\mathcal{C}_0$, written by definition as*

$$\varphi = \sum_{x \in \Lambda} \Big\{ \nabla_x (\mathrm{F}_x) + \nabla_{x,x+1}(\mathrm{G}_x) \Big\},$$

for some $\Lambda \subset \mathbb{Z}$ and some functions F_x and G_x satisfying the conditions of Definition 2.3. Let us consider $h \in \mathcal{C}$ with support in Λ_ℓ. Denote by ℓ_φ the integer $\ell_\varphi := \ell - s_\varphi - 1$ so that the supports of φ and its gradients $\nabla_{x,x+1}\varphi$ are included in Λ_ℓ for every $x \in \Lambda_\varphi$.

Then, there exists a positive constant $\mathrm{C}(\varphi,\gamma)$ which depends only on φ and γ such that

$$\left| \mathbb{E}^\star_\beta \Bigg[\sum_{|x| \leqslant \ell_\varphi} \tau_x \varphi, h \Bigg] \right| \leqslant \mathrm{C}(\varphi,\gamma) \, \big(\mathcal{D}_\ell(\mathbb{P}^\star_\beta; h)\big)^{1/2}. \tag{10}$$

Proof. Let us assume first that $\varphi = \nabla_0(\mathrm{F}_0)$, so that $s_\varphi = 1$. Then we have

$$\begin{aligned}\Big|\mathbb{E}^\star_\beta\Big[\sum_{|x|\leqslant \ell_\varphi} \tau_x\varphi, h\Big]\Big| = \Big|\sum_{|x|\leqslant \ell_\varphi} \mathbb{E}^\star_\beta\big[\tau_x \mathrm{F}_0, \nabla_x h\big]\Big| &\leqslant \sum_{|x|\leqslant \ell_\varphi} \mathbb{E}^\star_\beta\big[(\tau_x\mathrm{F}_0)^2\big]^{1/2}\, \mathbb{E}^\star_\beta\big[(\nabla_x h)^2\big]^{1/2} \\ &\leqslant \Big(\sum_{|x|\leqslant \ell_\varphi} \mathbb{E}^\star_\beta\big[(\tau_x\mathrm{F}_0)^2\big]\Big)^{1/2}\Big(\frac{2}{\gamma}\,\mathcal{D}_\ell(\mathbb{P}^\star_\beta;h)\Big)^{1/2} \\ &\leqslant \sqrt{2\gamma^{-1}}\; |2\ell_\varphi+1|^{1/2}\, \mathbb{E}^\star_\beta\big[\mathrm{F}_0^2\big]^{1/2}\, \big(\mathcal{D}_\ell(\mathbb{P}^\star_\beta;h)\big)^{1/2}.\end{aligned}$$

Above we used the Cauchy-Schwarz inequality twice, and the fact coming from (7) that

$$\sum_{|x|\leqslant \ell_\varphi} \mathbb{E}^\star_\beta\big[(\nabla_x h)^2\big] \leqslant \frac{2}{\gamma}\,\mathcal{D}_\ell(\mathbb{P}^\star_\beta;h).$$

In the same way, if $\varphi = \sum_{y\in\Lambda}\nabla_y(\mathrm{F}_y)$, we have

$$\begin{aligned}\Big|\mathbb{E}^\star_\beta\Big[\sum_{|x|\leqslant \ell_\varphi} \tau_x\varphi, h\Big]\Big| &= \Big|\sum_{|x|\leqslant \ell_\varphi}\sum_{y\in\Lambda} \mathbb{E}^\star_\beta\big[\tau_x\mathrm{F}_y, \nabla_{y+x}h\big]\Big| \\ &\leqslant \sum_{|x|\leqslant \ell_\varphi}\sum_{y\in\Lambda}\mathbb{E}^\star_\beta\big[(\tau_x\mathrm{F}_y)^2\big]^{1/2}\,\mathbb{E}^\star_\beta\big[(\nabla_{x+y}h)^2\big]^{1/2} \\ &\leqslant \sqrt{2\gamma^{-1}}(2s_\varphi+1)|2\ell_\varphi+1|^{1/2}\Big(\sup_{y\in\Lambda}\mathbb{E}^\star_\beta\big[\mathrm{F}_y^2\big]\Big)^{1/2}\big(\mathcal{D}_\ell(\mathbb{P}^\star_\beta;h)\big)^{1/2}.\end{aligned}$$

Therefore, the following constant

$$\mathrm{C}(\varphi,\gamma) := \sqrt{2\gamma^{-1}}(2s_\varphi+1)|2\ell_\varphi+1|^{1/2}\Big\{\Big(\sup_{x\in\Lambda}\mathbb{E}^\star_\beta\big[\mathrm{F}_x^2\big]\Big)^{1/2} + \Big(\sup_{x\in\Lambda}\mathbb{E}^\star_\beta\big[\mathrm{G}_x^2\big]\Big)^{1/2}\Big\}$$

satisfies (10). The general case easily follows. □

The main focus of this paper will be on the following quantities: for any $\varphi\in\mathcal{C}$ let us define

$$\mathbb{E}^\star_\beta\big[\varphi,(-\mathcal{S})^{-1}\varphi\big] = \sup_{h\in\mathcal{C}}\Big\{2\mathbb{E}^\star_\beta\big[\varphi,h\big] - \mathcal{D}_{s_h}(\mathbb{P}^\star_\beta;h)\Big\} \in [0,+\infty]. \tag{11}$$

By polarization, this definition can be extended to give a meaning to $\mathbb{E}^\star_\beta\big[\varphi,(-\mathcal{S})^{-1}\psi\big]$, for any $\varphi,\psi\in\mathcal{C}$. As a consequence of the previous results, these quantities are well defined for functions in $\mathcal{C}_0$:

Corollary 2.3. *For every function $\varphi\in\mathcal{C}_0$, the quantity $\mathbb{E}^\star_\beta[\varphi,(-\mathcal{S})^{-1}\varphi]$ is finite.*

Proof. This is a consequence of (10) and of the variational formula (11):

$$\begin{aligned}\mathbb{E}^\star_\beta\big[\varphi,(-\mathcal{S})^{-1}\varphi\big] &= \sup_{h\in\mathcal{C}}\Big\{2\mathbb{E}^\star_\beta\big[\varphi,h\big] - \mathcal{D}_{s_h}(\mathbb{P}^\star_\beta;h)\Big\} \\ &\leqslant \sup_{h\in\mathcal{C}}\Big\{\mathrm{C}(\varphi,\gamma)\big(\mathcal{D}_{s_h}(\mathbb{P}^\star_\beta;h)\big)^{1/2} - \mathcal{D}_{s_h}(\mathbb{P}^\star_\beta;h)\Big\} = \frac{\mathrm{C}^2(\varphi,\gamma)}{4} < \infty.\end{aligned}$$

□

Finally, if we use the decomposition of every function in $\mathbf{L}^2(\mu_\beta)$ over the basis of Hermite polynomials, we can prove the following result for functions in $\mathcal{Q}_0$ (the details for the proof are given in Appendix A, Proposition A.3.):

Proposition 2.4 (Variance of quadratic functions)**.** *If $\varphi\in\mathcal{Q}_0$, then*

$$\mathbb{E}^\star_\beta\big[\varphi,(-\mathcal{S}_{\Lambda_\varphi})^{-1}\varphi\big] = \sup_{\substack{g\in\mathcal{Q}\\ s_g=s_\varphi}}\Big\{2\,\mathbb{E}^\star_\beta\big[\varphi,g\big] - \mathcal{D}_{s_\varphi}(\mathbb{P}^\star_\beta;g)\Big\}.$$

2.5 Semi inner products and diffusion coefficient

For cylinder functions $g, h \in \mathcal{C}$, let us define:

$$\ll g, h \gg_{\beta,\star} := \sum_{x\in\mathbb{Z}} \mathbb{E}_\beta^\star[g\,\tau_x h], \qquad \text{and} \qquad \ll g \gg_{\beta,\star\star} := \sum_{x\in\mathbb{Z}} x\,\mathbb{E}_\beta^\star\big[g\,\omega_x^2\big]. \tag{12}$$

Both quantities are well defined because g and h belong to $\mathcal{C}$ and therefore all but a finite number of terms on each sum vanish.

REMARK 2.1. Note that $\ll \cdot,\cdot \gg_{\beta,\star}$ is a semi inner product, since the following equality holds:

$$\ll g, h \gg_{\beta,\star} = \lim_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|}\mathbb{E}_\beta^\star\Big[\sum_{x\in\Lambda}\tau_x g, \sum_{x\in\Lambda}\tau_x h\Big].$$

Since $\ll g - \tau_x g, h \gg_{\beta,\star} = 0$ for all $x \in \mathbb{Z}$, this inner product is not definite. In particular we have $\ll j_{0,1}^{\mathrm{S}}, h \gg_{\beta,\star} = 0$ for any $h \in \mathcal{C}$.

In the next proposition we give explicit formulas for elements of $\mathcal{C}_0$.

PROPOSITION 2.5. *If $\varphi \in \mathcal{C}_0$ with*

$$\varphi = \sum_{x\in\Lambda}\Big\{\nabla_x(\mathrm{F}_x) + \nabla_{x,x+1}(\mathrm{G}_x)\Big\},$$

then

$$\ll \varphi \gg_{\beta,\star\star} = \mathbb{E}_\beta^\star\Big[(\omega_0^2 - \omega_1^2)\sum_{x\in\Lambda}\tau_{-x}\mathrm{G}_x\Big],$$

$$\ll \varphi, g \gg_{\beta,\star} = \mathbb{E}_\beta^\star\Big[\nabla_0(\Gamma_g)\sum_{x\in\Lambda}\tau_{-x}\mathrm{F}_x + \nabla_{0,1}(\Gamma_g)\sum_{x\in\Lambda}\tau_{-x}\mathrm{G}_x\Big] \qquad \text{for all } g \in \mathcal{C}.$$

Proof. The proof is straightforward. □

We are now able to give the definition of the diffusion coefficient, which is going to be rigorously derived from the non-gradient approach detailed in the next sections.

DEFINITION 2.4. *We define the diffusion coefficient* $\mathrm{D}(\beta)$ *for* $\beta > 0$ *as*

$$\mathrm{D}(\beta) := \lambda + \frac{1}{\chi(\beta)}\inf_{f\in\mathcal{Q}}\sup_{g\in\mathcal{Q}}\Big\{\ll f, -\mathcal{S}f \gg_{\beta,\star} + 2 \ll j_{0,1}^{\mathrm{A}} - \mathcal{A}^{\mathbf{m}} f, g \gg_{\beta,\star} - \ll g, -\mathcal{S}g \gg_{\beta,\star}\Big\}. \tag{13}$$

The first term in the sum (λ) is only due to the exchange noise, whereas the second one comes from the hamiltonian part of the dynamics. Formally, this formula could be read as

$$\mathrm{D}(\beta) = \lambda + \frac{1}{\chi(\beta)} \ll j_{0,1}^{\mathrm{A}}, (-\mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^{\mathrm{A}} \gg_{\beta,\star}, \tag{14}$$

but the last term is not well defined because $j_{0,1}^{\mathrm{A}}$ is not in the range of $\mathcal{L}^{\mathbf{m}}$. More rigorously, we should define

$$\overline{\mathrm{D}}(\beta) := \lambda + \frac{1}{\chi(\beta)}\limsup_{z\to 0} \ll j_{0,1}^{\mathrm{A}}, (z - \mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^{\mathrm{A}} \gg_{\beta,\star}. \tag{15}$$

The last expression, called *Green-Kubo formula*, is now well defined, and the problem is reduced to prove convergence as $z \to 0$. In Section 7, we prove that (15) indeed converges (the proof being inspired by

[3]), and we also show that the diffusion coefficient can be equivalently defined in the two ways. Note that, assuming the convergence in (15), one can easily see that $\overline{\mathrm{D}}(\beta)$ does not depend on β. Let $\mathbf{L}^2_{\beta,\star}$ be the Hilbert space generated by the closure of $\{g \in \mathcal{C}\ ;\ \ll g, g \gg_{\beta,\star} < \infty\}$ w.r.t. the inner product $\ll \cdot \gg_{\beta,\star}$. Consider $h_z := h_z(\mathbf{m}, \omega; \beta)$ the solution to the resolvent equation in $\mathbf{L}^2_{\beta,\star}$ which reads as

$$(z - \mathcal{L}^{\mathbf{m}})h_z = j^{\mathrm{A}}_{0,1}.$$

Observe that if ω is distributed according to μ_β then $\beta^{1/2}\omega$ is distributed according to μ_1. Besides, $j^{\mathrm{A}}_{0,1}$ is a homogeneous polynomial of degree two in ω, which implies that h_z is also a homogeneous polynomial of degree two (by the fact that $\mathcal{L}^{\mathbf{m}}$ preserves every class of homogeneous polynomials). It follows that

$$\ll j^{\mathrm{A}}_{0,1}, (z - \mathcal{L}^{\mathbf{m}})^{-1} j^{\mathrm{A}}_{0,1} \gg_{\beta,\star} = \ll h_z, j^{\mathrm{A}}_{0,1} \gg_{\beta,\star} = \frac{1}{\beta^2} \ll h_z, j^{\mathrm{A}}_{0,1} \gg_{1,\star},$$

so that (15) in turns shows that $\overline{\mathrm{D}}$ does not depend on β, since by definition $\chi(\beta) = 2\beta^{-2}$.

3 Macroscopic fluctuations of energy

In this section we state our main result on the fluctuations of the empirical energy around equilibrium. We show that the limit fluctuation process is governed by a generalized Ornstein-Uhlenbeck process, whose covariances are given in terms of the diffusion coefficient given in Definition 2.4. For that purpose, we adapt the non-gradient method introduced by Varadhan. In particular, we rigorously establish the variational formula that appears in Definition 2.4. The non-gradient approach is detailed and split in several steps, in Sections 4, 5 and 6.

3.1 Energy fluctuation field

Recall that we denote by $\mathbf{e}_\beta$ the thermodynamical energy associated to $\beta > 0$, namely $\mathbf{e}_\beta = \beta^{-1}$. We define the energy empirical distribution $\pi^{\mathrm{N}}_{t,\mathbf{m}}$ on the continuous torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as

$$\pi^{\mathrm{N}}_{t,\mathbf{m}}(\mathrm{d}u) = \frac{1}{\mathrm{N}} \sum_{x \in \mathbb{T}_\mathrm{N}} \omega_x^2(t)\delta_{x/\mathrm{N}}(\mathrm{d}u), \quad t \in [0,\mathrm{T}], \quad u \in \mathbb{T},$$

where δ_u stands for the Dirac measure at point u, and where $\{\omega(t)\}_{t\geqslant 0}$ is the Markov process generated by $\mathrm{N}^2\mathcal{L}^{\mathbf{m}}_{\mathrm{N}}$. If the initial state of the dynamics is the equilibrium Gibbs measure μ^{N}_β, then, for any fixed $t \geqslant 0$, and any disorder $\mathbf{m} \in \Omega_{\mathcal{D}}$, the measure $\pi^{\mathrm{N}}_{t,\mathbf{m}}$ weakly converges towards the measure $\{\mathbf{e}_\beta \mathrm{d}u\}$ on $\mathbb{T}$, which is deterministic and with constant density w.r.t. the Lebesgue measure on $\mathbb{T}$. Here we investigate the fluctuations of the empirical measure $\pi^{\mathrm{N}}_{t,\mathbf{m}}$ with respect to this limit.

Definition 3.1 (Energy fluctuation field). *We denote by $\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}$ the empirical energy fluctuation field associated with the Markov process $\{\omega(t)\}_{t\geqslant 0}$ generated by $\mathrm{N}^2\mathcal{L}^{\mathbf{m}}_{\mathrm{N}}$ and starting from $\mathbb{P}^\star_\beta = \mathbb{P} \otimes \mu^{\mathrm{N}}_\beta$, defined by its action over test functions $\mathrm{H} \in \mathrm{C}^2(\mathbb{T})$,*

$$\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}(\mathrm{H}) = \frac{1}{\sqrt{\mathrm{N}}} \sum_{x \in \mathbb{T}_\mathrm{N}} \mathrm{H}\left(\frac{x}{\mathrm{N}}\right) \left(\omega_x^2(t) - \mathbf{e}_\beta\right).$$

We are going to prove that the annealed distribution of $\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}$ converges in distribution towards the solution to the linear SPDE:

$$\partial_t \mathcal{Y} = \mathrm{D}\, \partial_y^2 \mathcal{Y} + \sqrt{2\mathrm{D}\chi(\beta)}\, \partial_y \mathrm{W}, \tag{16}$$

where W is a standard normalized space-time white noise, and D is the diffusion coefficient defined in Definition 2.4. More precisely, the solution to (16) is the stationary generalized Ornstein-Uhlenbeck process with zero mean and covariances given by

$$\mathbb{E}_{\mathcal{Y}}\left[\mathcal{Y}_t(\mathrm{H})\mathcal{Y}_0(\mathrm{G})\right] = \frac{\chi(\beta)}{\sqrt{4\pi t\mathrm{D}}}\int_{\mathbb{R}^2} \overline{\mathrm{H}}(u)\overline{\mathrm{G}}(v)\exp\left(-\frac{(u-v)^2}{4t\mathrm{D}}\right)\mathrm{d}u\,\mathrm{d}v,$$

for all $t \geqslant 0$ and test functions $\mathrm{H},\mathrm{G} \in \mathrm{C}^2(\mathbb{T})$. Here and after, $\overline{\mathrm{H}}$ (resp. $\overline{\mathrm{G}}$) is the periodic extension to the real line of H (resp. G). Let us fix a time horizon $\mathrm{T} > 0$. The probability measure on the Skorokhod space $\mathcal{D}([0,\mathrm{T}],\Omega^{\mathrm{N}})$ induced by the Markov process $\{\omega(t)\}_{t\geqslant 0}$ generated by $\mathrm{N}^2\mathcal{L}_{\mathrm{N}}^{\mathbf{m}}$ and starting from $\mathbb{P}^\star_\beta = \mathbb{P}\otimes\mu_\beta^{\mathrm{N}}$ is denoted by $\mathbb{P}^\star_{\mu_\beta^{\mathrm{N}}}$. Expectation with respect to $\mathbb{P}^\star_{\mu_\beta^{\mathrm{N}}}$ is denoted by $\mathbb{E}^\star_{\mu_\beta^{\mathrm{N}}}$.

Consider for $k > 5/2$ the Sobolev space $\mathfrak{H}_{-k}$ of distributions $\mathcal{Y}$ on $\mathbb{T}$ with finite norm

$$\|\mathcal{Y}\|_{-k}^2 = \sum_{n\geqslant 1}(\pi n)^{-2k}\left|\mathcal{Y}(e_n)\right|^2,$$

where e_n is the function $x \mapsto \sqrt{2}\sin(\pi n x)$. We denote by $\mathfrak{Y}^{\mathrm{N}}$ the annealed probability measure on the space $\mathcal{D}([0,\mathrm{T}],\mathfrak{H}_{-k})$ of continuous trajectories on the Sobolev space, induced by the Markov process $\{\omega(t)\}_{t\geqslant 0}$ and the mapping $\mathcal{Y}^{\mathrm{N}} : (\mathbf{m},\omega) \mapsto \{\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}\}_{0\leqslant t\leqslant \mathrm{T}}$. In other words, we define

$$\mathfrak{Y}^{\mathrm{N}}(\mathcal{Y}\in\cdot) = \mathbb{P}^\star_{\mu_\beta^{\mathrm{N}}} \circ (\mathcal{Y}^{\mathrm{N}})^{-1}(\,\cdot\,).$$

Finally, we let $\mathfrak{Y}$ be the probability measure on the space $\mathcal{C}([0,\mathrm{T}],\mathfrak{H}_{-k})$ corresponding to the generalized Ornstein-Uhlenbeck process $\mathcal{Y}_t$ solution to (16). The main result of this section is the following.

Theorem 3.1. *Fix $k > 5/2$ and $\mathrm{T} > 0$. The sequence $\{\mathfrak{Y}^{\mathrm{N}}\}_{\mathrm{N}\geqslant 1}$ weakly converges in $\mathcal{D}([0,\mathrm{T}],\mathfrak{H}_{-k})$ to the probability measure $\mathfrak{Y}$.*

3.2 Strategy of the proof

We follow the lines of [23, Section 3]. The proof of Theorem 3.1 is divided into three steps. First, we need to show that the sequence $\{\mathfrak{Y}^{\mathrm{N}}\}_{\mathrm{N}\geqslant 1}$ is tight. This point follows a standard argument, given for instance in [16, Section 11], and recalled in Appendix C for the sake of completeness. Then, we prove that any limit point $\mathfrak{Y}^\star$ of $\{\mathfrak{Y}^{\mathrm{N}}\}_{\mathrm{N}\geqslant 1}$ is concentrated on trajectories whose marginals at time t have, for any $t \in [0,\mathrm{T}]$, the distribution of a centered Gaussian field with covariances given by

$$\mathfrak{Y}^\star\left[\mathcal{Y}_t(\mathrm{H})\mathcal{Y}_t(\mathrm{G})\right] = \chi(\beta)\int_{\mathbb{T}} \mathrm{H}(u)\mathrm{G}(u)\,\mathrm{d}u,$$

where $\mathrm{H},\mathrm{G} \in \mathrm{C}^2(\mathbb{T})$ are test functions. Since μ_β^{N} is stationary for the process ω, this statement comes from the central limit theorem for independent variables. Finally, we prove the main point in the next subsections: all limit points $\mathfrak{Y}^\star$ of the sequence $\{\mathfrak{Y}^{\mathrm{N}}\}_{\mathrm{N}\geqslant 1}$ solve the martingale problems (17) and (18) given below, namely, for any function $\mathrm{H} \in \mathrm{C}^2(\mathbb{T})$,

$$\mathfrak{M}_t(\mathrm{H}) := \mathcal{Y}_t(\mathrm{H}) - \mathcal{Y}_0(\mathrm{H}) - \int_0^t \mathrm{D}\mathcal{Y}_s(\mathrm{H}'')\mathrm{d}s, \tag{17}$$

and

$$\mathfrak{N}_t(\mathrm{H}) := (\mathfrak{M}_t(\mathrm{H}))^2 - 2t\chi(\beta)\mathrm{D}\int_{\mathbb{T}} \mathrm{H}'(u)^2\mathrm{d}u \tag{18}$$

are $\mathbf{L}^1(\mathfrak{Y}^\star)$-martingales.

3.3 Martingale decompositions

In what follows, in order to simplify notation we write $f(\mathbf{m},s) := f(\mathbf{m},\omega(s))$ for any f which is defined on $\Omega_{\mathcal{D}} \times \Omega_{\mathrm{N}}$. Let us fix $\mathrm{H} \in \mathrm{C}^2(\mathbb{T})$ and $t \in [0,\mathrm{T}]$. Itô calculus, and a discrete integration by parts, permits to decompose $\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}(\mathrm{H})$ as

$$\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}(\mathrm{H}) = \mathcal{Y}^{\mathrm{N}}_{0,\mathbf{m}}(\mathrm{H}) + \int_0^t \sqrt{\mathrm{N}} \sum_{x\in\mathbb{T}_\mathrm{N}} \nabla_\mathrm{N}\mathrm{H}\left(\frac{x}{\mathrm{N}}\right) j_{x,x+1}(\mathbf{m},s)\,\mathrm{d}s + \mathcal{M}^{\mathrm{N}}_{t,\mathbf{m}}(\mathrm{H}) \tag{19}$$

where $\mathcal{M}^{\mathrm{N}}_{t,\mathbf{m}}$ is the martingale defined as

$$\mathcal{M}^{\mathrm{N}}_{t,\mathbf{m}}(\mathrm{H}) = \int_0^t \frac{1}{\mathrm{N}\sqrt{\mathrm{N}}} \sum_{x\in\mathbb{T}_\mathrm{N}} \nabla_\mathrm{N}\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big)\big(\omega_x^2-\omega_{x+1}^2\big)(s)\,\mathrm{d}\big[\mathrm{N}_{x,x+1}(\lambda\mathrm{N}^2 s)-\lambda\mathrm{N}^2 s\big].$$

Here and after, $\{\mathrm{N}_{x,x+1}(t)\}_{x\in\mathbb{Z},t\geqslant 0}$ and $\{\mathrm{N}_x(t)\}_{x\in\mathbb{Z},t\geqslant 0}$ are independent Poisson processes of intensity 1, and ∇_N stands for the discrete gradient:

$$\nabla_\mathrm{N}\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big) = \mathrm{N}\left[\mathrm{H}\Big(\frac{x+1}{\mathrm{N}}\Big)-\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big)\right].$$

In what follows, the discrete Laplacian Δ_N is defined in a similar way:

$$\Delta_\mathrm{N}\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big) = \mathrm{N}^2\left[\mathrm{H}\Big(\frac{x+1}{\mathrm{N}}\Big)+\mathrm{H}\Big(\frac{x-1}{\mathrm{N}}\Big)-2\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big)\right].$$

To close the equation, we are going to replace the term involving the microscopic currents in (19) with a term involving $\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}$. In other words, the dominant contribution in

$$\int_0^t \sqrt{\mathrm{N}} \sum_{x\in\mathbb{T}_\mathrm{N}} \nabla_\mathrm{N}\mathrm{H}\left(\frac{x}{\mathrm{N}}\right) j_{x,x+1}(\mathbf{m},s)\,\mathrm{d}s$$

is its projection over the conservation field $\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}$ (recall that the total energy is the unique conserved quantity of the system). The non-gradient approach consists in using the *fluctuation-dissipation approximation* of the current $-j_{x,x+1}$ as $\mathrm{D}\big(\omega_{x+1}^2-\omega_x^2\big)+\mathcal{L}^{\mathbf{m}}(\tau_x f)$. This replacement is made rigorous in Theorem 5.9 below.

After adding and subtracting $\mathrm{D}\big(\omega_{x+1}^2-\omega_x^2\big)+\mathcal{L}^{\mathbf{m}}(\tau_x f)$ in (19) above, we can rewrite it, for any $f\in\mathcal{Q}$, as follows:

$$\mathcal{Y}^{\mathrm{N}}_{t,\mathbf{m}}(\mathrm{H}) = \mathcal{Y}^{\mathrm{N}}_{0,\mathbf{m}}(\mathrm{H}) + \int_0^t \mathrm{D}\mathcal{Y}^{\mathrm{N}}_{s,\mathbf{m}}(\Delta_\mathrm{N}\mathrm{H})\mathrm{d}s + \mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) + \mathfrak{I}^{2,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) + \mathfrak{M}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) + \mathfrak{M}^{2,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}), \tag{20}$$

where

$$\mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) = \int_0^t \sqrt{\mathrm{N}} \sum_{x\in\mathbb{T}_\mathrm{N}} \nabla_\mathrm{N}\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big)\Big[j_{x,x+1}(\mathbf{m},s) + \mathrm{D}\big(\omega_{x+1}^2-\omega_x^2\big)(s) + \mathcal{L}^{\mathbf{m}}(\tau_x f)(\mathbf{m},s)\Big]\,\mathrm{d}s,$$

$$\mathfrak{I}^{2,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) = -\int_0^t \sqrt{\mathrm{N}} \sum_{x\in\mathbb{T}_\mathrm{N}} \nabla_\mathrm{N}\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big)\mathcal{L}^{\mathbf{m}}(\tau_x f)(\mathbf{m},s)\mathrm{d}s,$$

$$\mathfrak{M}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) = -\int_0^t \frac{1}{\mathrm{N}\sqrt{\mathrm{N}}}\sum_{x\in\mathbb{T}_\mathrm{N}} \nabla_\mathrm{N}\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big)\Big\{\big[\nabla_{x,x+1}(\omega_x^2-\Gamma_f)\big](s)\,\mathrm{d}\big[\mathrm{N}_{x,x+1}(\lambda\mathrm{N}^2 s)-\lambda\mathrm{N}^2 s\big] \\ - \nabla_x(\Gamma_f)(s)\,\mathrm{d}\big[\mathrm{N}_x(\gamma\mathrm{N}^2 s)-\gamma\mathrm{N}^2 s\big]\Big\},$$

$$\mathfrak{M}^{2,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) = -\int_0^t \frac{1}{\mathrm{N}\sqrt{\mathrm{N}}}\sum_{x\in\mathbb{T}_\mathrm{N}} \nabla_\mathrm{N}\mathrm{H}\Big(\frac{x}{\mathrm{N}}\Big)\Big\{\nabla_{x,x+1}(\Gamma_f)(s)\,\mathrm{d}\big[\mathrm{N}_{x,x+1}(\lambda\mathrm{N}^2 s)-\lambda\mathrm{N}^2 s\big] \\ + \nabla_x(\Gamma_f)(s)\,\mathrm{d}\big[\mathrm{N}_x(\gamma\mathrm{N}^2 s)-\gamma\mathrm{N}^2 s\big]\Big\}.$$

The proof is based on the following two results.

Lemma 3.2. *For every function $\mathrm{H}\in\mathrm{C}^2(\mathbb{T})$, and every function $f\in\mathcal{Q}$,*

$$\lim_{\mathrm{N}\to\infty}\mathbb{E}^{\star}_{\mu^\mathrm{N}_\beta}\Big[\sup_{0\leqslant t\leqslant \mathrm{T}}\Big(\mathfrak{I}^{2,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H})+\mathfrak{M}^{2,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H})\Big)^2\Big]=0.$$

Theorem 3.3 (Boltzmann-Gibbs principle)**.** *There exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}}\in\mathcal{Q}$ such that*

(i) for every function $\mathrm{H}\in\mathrm{C}^2(\mathbb{T})$,

$$\lim_{k\to\infty}\lim_{\mathrm{N}\to\infty}\mathbb{E}^{\star}_{\mu^\mathrm{N}_\beta}\Big[\sup_{0\leqslant t\leqslant \mathrm{T}}\Big(\mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f_k}(\mathrm{H})\Big)^2\Big]=0, \tag{21}$$

(ii) and moreover

$$\lim_{k\to\infty}\mathbb{E}^{\star}_{\beta}\Big[\lambda\Big(\nabla_{0,1}(\omega_0^2-\Gamma_{f_k})\Big)^2+\gamma\Big(\nabla_0(\Gamma_{f_k})\Big)^2\Big]=2\mathrm{D}\chi(\beta). \tag{22}$$

Remark 3.1. Note that the expectation at the left hand side of (22) also rewrites as

$$2\lambda\chi(\beta)+\mathbb{E}^{\star}_{\beta}\Big[\lambda\big(\nabla_{0,1}(\Gamma_{f_k})\big)^2+\gamma\big(\nabla_0(\Gamma_{f_k})^2\Big],$$

since for any $f\in\mathcal{C}$, one can check that $\mathbb{E}^{\star}_{\beta}\big[(\omega_0^2-\omega_1^2)\nabla_{0,1}\Gamma_f\big]=0$.

Using analogous ingredients as in Lemma 3.2 and Theorem 3.3, it is then straightforward to prove that the martingale $\mathfrak{M}^{1,\mathrm{N}}_{t,\mathbf{m},f_k}$ converges in $\mathbf{L}^2(\mathbb{P}^{\star}_{\beta})$, as $\mathrm{N}\to\infty$ then $k\to\infty$, to a martingale $\mathfrak{M}_t(\mathrm{H})$ of quadratic variation

$$2t\mathrm{D}\chi(\beta)\int_{\mathbb{T}}\mathrm{H}'(u)^2\,\mathrm{d}u,$$

and the limit $\mathcal{Y}_t(\mathrm{H})$ of $\mathcal{Y}^\mathrm{N}_{t,\mathbf{m}}(\mathrm{H})$ satisfies the equation

$$\mathcal{Y}_t(\mathrm{H})=\mathcal{Y}_0(\mathrm{H})+\int_0^t\mathcal{Y}_s(\mathrm{D}\mathrm{H}'')\,\mathrm{d}s+\mathfrak{M}_t(\mathrm{H}).$$

therefore any limit point $\mathfrak{Y}^\star$ of the sequence $\{\mathfrak{Y}^\mathrm{N}\}_{\mathrm{N}\geqslant 1}$ is concentrated on trajectories $\mathcal{Y}$ solving the martingale problems (17) and (18), which uniquely characterized the generalized Ornstein-Uhlenbeck process $\mathcal{Y}_t$. The proof of Lemma 3.2 is the content of the next subsection. The proof of Theorem 3.3 is more challenging, and Sections 4, 5 and 6 are devoted to it.

3.4 Proof of Lemma 3.2

In this paragraph we give a proof of Lemma 3.2. We define for any $f \in \mathcal{Q}$

$$X^N_{t,\mathbf{m},f}(H) = -\frac{1}{N\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \tau_x f(\mathbf{m},t)$$

First, by rewriting (19) with $X^N_{t,\mathbf{m},f}(H)$ instead of $\mathcal{Y}^N_{t,\mathbf{m}}(H)$, one straightforwardly obtains

$$\begin{aligned}\mathfrak{I}^{2,N}_{t,\mathbf{m},f}(H) + \mathfrak{M}^{2,N}_{t,\mathbf{m},f}(H) &= X^N_{t,\mathbf{m},f}(H) - X^N_{0,\mathbf{m},f}(H)\\ &+ \frac{1}{N\sqrt{N}}\int_0^t \sum_{x\in\mathbb{T}_N} \nabla_{x,x+1}\Big(\sum_{z\in\mathbb{T}_N} \nabla_N H\Big(\frac{z}{N}\Big)\tau_z f - \nabla_N H\Big(\frac{x}{N}\Big)\Gamma_f\Big)(\mathbf{m},s)\mathrm{d}\big[N_{x,x+1}(\lambda N^2 s) - \lambda N^2 s\big]\\ &+ \frac{1}{N\sqrt{N}}\int_0^t \sum_{x\in\mathbb{T}_N} \nabla_{x}\Big(\sum_{z\in\mathbb{T}_N} \nabla_N H\Big(\frac{z}{N}\Big)\tau_z f - \nabla_N H\Big(\frac{x}{N}\Big)\Gamma_f\Big)(\mathbf{m},s)\mathrm{d}\big[N_{x}(\gamma N^2 s) - \gamma N^2 s\big].\end{aligned}$$

Therefore, using the convexity inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, we obtain

$$\begin{aligned}&\Big(\mathfrak{I}^{2,N}_{t,\mathbf{m},f}(H) + \mathfrak{M}^{2,N}_{t,\mathbf{m},f}(H)\Big)^2 \leqslant 3\Big(X^N_{t,\mathbf{m},f}(H) - X^N_{0,\mathbf{m},f}(H)\Big)^2\\ &+3\Big(\frac{1}{N\sqrt{N}}\int_0^t \sum_{x\in\mathbb{T}_N} \nabla_{x,x+1}\Big(\sum_{z\in\mathbb{T}_N} \nabla_N H\Big(\frac{z}{N}\Big)\tau_z f - \nabla_N H\Big(\frac{x}{N}\Big)\Gamma_f\Big)(\mathbf{m},s)\mathrm{d}\big[N_{x,x+1}(\lambda N^2 s) - \lambda N^2 s\big]\Big)^2 \quad (23)\\ &+3\Big(\frac{1}{N\sqrt{N}}\int_0^t \sum_{x\in\mathbb{T}_N} \nabla_{x}\Big(\sum_{z\in\mathbb{T}_N} \nabla_N H\Big(\frac{z}{N}\Big)\tau_z f - \nabla_N H\Big(\frac{x}{N}\Big)\Gamma_f\Big)(\mathbf{m},s)\mathrm{d}\big[N_{x}(\gamma N^2 s) - \gamma N^2 s\big]\Big)^2. \quad (24)\end{aligned}$$

On the one hand, for any $t \in [0,T]$

$$\mathbb{E}^\star_\beta\Big[\big(X^N_{t,\mathbf{m},f}(H)\big)^2\Big] = \frac{1}{N^3}\sum_{x,y\in\mathbb{T}_N} \nabla_N H\Big(\frac{x}{N}\Big)\nabla_N H\Big(\frac{y}{N}\Big)\mathbb{E}^\star_\beta\big[\tau_x f, \tau_y f\big].$$

This last quantity is of order $1/N^2$, because f is a local function with mean zero, and H is smooth. On the other hand, let us define,

$$Y_x(\mathbf{m},\omega) := \sum_{z\in\mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right)\tau_z f - \nabla_N H\left(\frac{x}{N}\right)\sum_{z\in\mathbb{Z}}\tau_z f,$$

which is ill defined, but for which

$$\nabla_{x,x+1} Y_x(\mathbf{m},\omega) := \sum_{|z-x|\leqslant \ell_f+1}\left[\nabla_N H\left(\frac{z}{N}\right) - \nabla_N H\left(\frac{x}{N}\right)\right]\tau_z f$$

is not. Moreover, the $\mathbf{L}^2(\mathbb{P}^\star_\beta)$–norm of $\nabla_{x,x+1}Y_x$ is of order $C(f)/N$ because H is assumed to be of class C^2: this implies that the expectation of (23) w.r.t. $\mathbb{E}^\star_\beta$ is

$$\frac{3\lambda^2 t N^2}{N^3}\sum_{x\in\mathbb{T}_N}\mathbb{E}^\star_\beta\Big[\big(\nabla_{x,x+1}(Y_x)\big)^2\Big] = \mathcal{O}(N^{-2}).$$

The same holds for (24).

4 CLT variances at equilibrium

In this section we are going to identify the diffusion coefficient D that appears in (20). Roughly speaking, D can be viewed as the asymptotic component of the energy current $j_{x,x+1}$ in the direction of the gradient $-(\omega_{x+1}^2-\omega_x^2)$, which makes the expression below vanish for any fixed $t \geqslant 0$

$$\inf_{f\in\mathcal{Q}} \limsup_{N\to\infty} \mathbb{E}^{\star}_{\mu_\beta^N}\left[\left|\int_0^t \sum_{x\in\mathbb{T}_N}\big[j_{x,x+1}+D(\omega_{x+1}^2-\omega_x^2)+\mathcal{L}^{\mathbf{m}}(\tau_x f)\big]ds\right|\right], \quad \text{for any } \beta>0.$$

Let us start by giving some known tools that will help understand the forthcoming results, at least at an informal level.

4.1 An insight through additive functionals of Markov processes

Consider a continuous time Markov process $\{Y_s\}_{s\geqslant 0}$ on a complete and separable metric space E, and admitting an invariant measure π. We denote by $\langle\cdot,\cdot\rangle_\pi$ the inner product in $\mathbf{L}^2(\pi)$ and by $\mathcal{L}$ the infinitesimal generator of the process. The adjoint of $\mathcal{L}$ in $\mathbf{L}^2(\pi)$ is denoted by $\mathcal{L}^\star$. Fix a function $V: E\to\mathbb{R}$ in $\mathbf{L}^2(\pi)$ such that $\langle V\rangle_\pi = 0$. Theorem 2.7 in [18] gives conditions on V which guarantee a central limit theorem for

$$\frac{1}{\sqrt{t}}\int_0^t V(Y_s)ds$$

and shows that the limiting variance equals

$$\sigma^2(V,\pi) = 2\lim_{\substack{z\to 0\\ z>0}} \langle V,(z-\mathcal{L})^{-1}V\rangle_\pi.$$

Let the generator $\mathcal{L}$ be decomposed as $\mathcal{L}=\mathcal{S}+\mathcal{A}$, where $\mathcal{S}=(\mathcal{L}+\mathcal{L}^\star)/2$ and $\mathcal{A}=(\mathcal{L}-\mathcal{L}^\star)/2$ are respectively the symmetric and antisymmetric parts of $\mathcal{L}$. Let $\mathcal{H}_1$ be the completion of the quotient of $\mathbf{L}^2(\pi)$ with respect to constant functions, for the semi-norm $\|\cdot\|_1$ defined as:

$$\|f\|_1^2 := \big\langle f,(-\mathcal{L})f\big\rangle_\pi = \big\langle f,(-\mathcal{S})f\big\rangle_\pi.$$

Let $\mathcal{H}_{-1}$ be the dual space of $\mathcal{H}_1$ with respect to $\mathbf{L}^2(\pi)$, in other words, the Hilbert space endowed with the norm $\|\cdot\|_{-1}$ defined by

$$\|f\|_{-1}^2 := \sup_g \left\{2\big\langle f,g\big\rangle_\pi - \|g\|_1^2\right\},$$

where the supremum is carried over some suitable set of functions g. Formally, $\|f\|_{-1}$ can also be thought as

$$\big\langle f,(-\mathcal{S})^{-1}f\big\rangle_\pi.$$

Note the difference with the variance $\sigma^2(V,\pi)$ which formally reads

$$2\big\langle V,(-\mathcal{L})^{-1}V\big\rangle_\pi = 2\big\langle V,\big[(-\mathcal{L})^{-1}\big]_s V\big\rangle_\pi.$$

Hereafter, B_s represents the symmetric part of the operator B. We can write, at least formally, that

$$\left\{\big[(-\mathcal{L})^{-1}\big]_s\right\}^{-1} = -\mathcal{S}+\mathcal{A}^\star(-\mathcal{S})^{-1}\mathcal{A} \geqslant -\mathcal{S},$$

where $\mathcal{A}^\star$ stands for the adjoint of $\mathcal{A}$. We have therefore that $\big[(-\mathcal{L})^{-1}\big]_s \leqslant (-\mathcal{S})^{-1}$. The following result is a rigorous estimate of the variance in terms of the $\mathcal{H}_{-1}$ norm, which is proved in [18, Lemma 2.4].

Lemma 4.1. *Given* $\mathrm{T} > 0$ *and a mean zero function* V *in* $\mathbf{L}^2(\pi) \cap \mathcal{H}_{-1}$,

$$\mathbb{E}_\pi\left[\sup_{0\leqslant t\leqslant \mathrm{T}}\left(\int_0^t \mathrm{V}(s)\mathrm{d}s\right)^2\right] \leqslant 24\mathrm{T}\|\mathrm{V}\|_{-1}^2. \tag{25}$$

In our case, the fact that the symmetric part of the generator does not depend on the disorder implies that (25) still holds if we take the expectation with respect to the disorder $\mathbb{P}$, and thus replace π with $\mathbb{P}\otimes\pi$. If we compare the previous left hand side to the Boltzmann-Gibbs principle (22), the next step should be to take V proportional to

$$\sum_{x\in\mathbb{T}_\mathrm{N}}\left[j_{x,x+1} + \mathrm{D}(\omega_{x+1}^2 - \omega_x^2) + \mathcal{L}^{\mathbf{m}}(\tau_x f)\right] \tag{26}$$

and then take the limit as N goes to infinity. In the right hand side of (25) we will obtain a variance that depends on N, and the main task will be to show that this variance converges: this is studied in more details in what follows. Precisely, we prove that the limit of the variance results in a semi-norm, which is denoted by $|||\cdot|||_\beta$ and defined in (27) below. More explicitly, we are going to see that (27) involves a variational formula, which formally reads

$$|||\varphi|||_\beta^2 = \ll \varphi, (-\mathcal{S})^{-1}\varphi \gg_{\beta,\star} + \frac{1}{\lambda\,\chi(\beta)} \ll \varphi \gg_{\beta,\star\star}^2 .$$

The final step consists in minimizing this semi-norm on a well-chosen subspace in order to get the Boltzmann-Gibbs principle, through orthogonal projections in Hilbert spaces. One significant difficulty is that $|||\cdot|||_\beta$ only depends on the symmetric part of the generator $\mathcal{S}$, and the latter is really degenerate, since it does not have a spectral gap.

In Subsection 4.2, we relate the previous limiting variance (which is obtained by taking the limit as N goes to infinity) to the suitable semi-norm. Subsection 4.3 is devoted to proving the Boltzmann-Gibbs principle (using Lemma 4.1). Note that (26) is a sum of local functions in $\mathcal{Q}_0$, from Proposition 2.1 (recall that $f\in\mathcal{Q}$). Therefore, all our results will be restricted to that subspace. Then, in Section 5 we investigate the Hilbert space generated by the semi-norm, and prove decompositions into direct sums. Finally, Section 6 focuses on the diffusion coefficient and its different expressions. These three main steps are quite standard, and many of the arguments can be found in [23]. For that reason, we shall be more brief in the exposition, and refer the reader to [23] for more details.

4.2 Limiting variance and semi-norm

We now assume $\beta = 1$. All statements are valid for any $\beta > 0$, and the general argument can be easily written. In the following, we deliberately keep the notation $\chi(1)$, even if the latter could be replaced with its exact value $\chi(1) = 2$. We are going to obtain a variational formula for the variance

$$\frac{1}{2\ell}\mathbb{E}_1^\star\Big[(-\mathcal{S}_{\Lambda_\ell})^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi, \sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi\Big]$$

where $\varphi\in\mathcal{Q}_0$ and $\ell_\varphi = \ell - s_\varphi - 1$. We first introduce a semi-norm on $\mathcal{Q}_0$:

DEFINITION 4.1. *For any cylinder function φ in $\mathcal{Q}_0$, let us define*

$$\left\vert\!\left\vert\!\left\vert \varphi \right\vert\!\right\vert\!\right\vert_1^2 := \sup_{g\in\mathcal{Q}} \Big\{ 2 \ll \varphi, g \gg_{1,\star} - \frac{\gamma}{2}\mathbb{E}_1^\star\Big[\big(\nabla_0\Gamma_g\big)^2\Big] - \frac{\lambda}{2}\mathbb{E}_1^\star\Big[\big(\nabla_{0,1}\Gamma_g\big)^2\Big]\Big\} + \frac{1}{\lambda\chi(1)} \ll \varphi \gg_{1,\star\star}^2 \tag{27}$$

$$= \sup_{\substack{g\in\mathcal{Q}\\ a\in\mathbb{R}}} \Big\{ 2 \ll \varphi, g \gg_{1,\star} + 2a \ll \varphi \gg_{1,\star\star} - \frac{\gamma}{2}\mathbb{E}_1^\star\Big[\big(\nabla_0\Gamma_g\big)^2\Big] - \frac{\lambda}{2}\mathbb{E}_1^\star\Big[\big(a(\omega_0^2-\omega_1^2) + \nabla_{0,1}\Gamma_g\big)^2\Big]\Big\}, \tag{28}$$

where $\ll \cdot \gg_{1,\star}$ and $\ll \cdot \gg_{1,\star\star}$ were introduced in (12).

REMARK 4.1. The second identity in (28) follows from an explicit computation of the supremum in $a \in \mathbb{R}$, which can be obtained by standard arguments, using the fact that $\mathbb{E}_1^\star\big[(\omega_0^2-\omega_1^2)\nabla_{0,1}\Gamma_g\big] = 0$ for any $g \in \mathcal{C}$.

Note that, from Proposition 2.5, one can easily bound $\left\vert\!\left\vert\!\left\vert \varphi \right\vert\!\right\vert\!\right\vert_1^2$ for any $\varphi \in \mathcal{Q}_0$ as follows: if

$$\varphi = \sum_{x\in\Lambda} \Big\{ \nabla_x(\mathrm{F}_x) + \nabla_{x,x+1}(\mathrm{G}_x) \Big\},$$

then

$$\left\vert\!\left\vert\!\left\vert \varphi \right\vert\!\right\vert\!\right\vert_1^2 \leqslant \frac{2}{\gamma}\mathbb{E}_1^\star\Big[\Big(\sum_{x\in\Lambda}\tau_{-x}\mathrm{F}_x\Big)^2\Big] + \frac{3}{\lambda}\mathbb{E}_1^\star\Big[\Big(\sum_{x\in\Lambda}\tau_{-x}\mathrm{G}_x\Big)^2\Big] < \infty.$$

We are now in position to state the main result of this subsection.

THEOREM 4.2. *Consider a quadratic function $\varphi \in \mathcal{Q}_0$. Then*

$$\lim_{\ell\to\infty} (2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1} \sum_{|x|\leqslant\ell_\varphi} \tau_x\varphi, \sum_{|x|\leqslant\ell_\varphi} \tau_x\varphi\Big] = \left\vert\!\left\vert\!\left\vert \varphi \right\vert\!\right\vert\!\right\vert_1^2.$$

Here, ℓ_φ stands for $\ell - s_\varphi - 1$ so that the support of $\tau_x\varphi$ is included in Λ_ℓ for every $x \in \Lambda_{\ell_\varphi}$.

This theorem is the key of the standard non-gradient method. As usual, the proof is done in two steps that we separate as two different lemmas for the sake of clarity. First, we bound the variance of a cylinder function $\varphi \in \mathcal{Q}_0$, with respect to $\mathbb{P}_1^\star$, by the semi-norm $\left\vert\!\left\vert\!\left\vert \varphi \right\vert\!\right\vert\!\right\vert_1^2$ (Lemma 4.3). In the second step, a lower bound for the variance can be easily deduced from the variational formula which expresses the variance as a supremum (11).

LEMMA 4.3. *Under the assumptions of Theorem 4.2,*

$$\limsup_{\ell\to\infty} (2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1} \sum_{|x|\leqslant\ell_\varphi} \tau_x\varphi, \sum_{|x|\leqslant\ell_\varphi} \tau_x\varphi\Big] \leqslant \left\vert\!\left\vert\!\left\vert \varphi \right\vert\!\right\vert\!\right\vert_1^2.$$

In the proof of this lemma, one needs to know the weak limits of some particular sequences in $\mathcal{Q}_0$. In the typical approach, these weak limits are viewed as *germs of closed forms*, but for the harmonic chain, this way of thinking is not necessary: this is one of the main technical novelties in this work. The rest of this section is devoted to proving Lemma 4.3.

Let us start by following the proof given in [23, Lemma 4.3] and we assume for the sake of clarity that $\varphi = \nabla_0(\mathrm{F}) + \nabla_{0,1}(\mathrm{G})$, for two quadratic cylinder functions F, G (the general case can then be deduced quite easily). We write the variational formula

$$(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi,\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi\Big] = \sup_{h\in\mathcal{C}}\Big\{2\mathbb{E}_1^\star\Big[\varphi,\frac{1}{2\ell}\sum_{|x|\leqslant\ell_\varphi}\tau_x h\Big] - \frac{1}{2\ell}\mathcal{D}_\ell(\mathbb{P}_1^\star;h)\Big\}$$
$$= \sup_{h\in\mathcal{C}}\Big\{2\mathbb{E}_1^\star\Big[\mathrm{F}\nabla_0\Big(\frac{1}{2\ell}\sum_{|x|\leqslant\ell_\varphi}\tau_x h\Big) + \mathrm{G}\nabla_{0,1}\Big(\frac{1}{2\ell}\sum_{|x|\leqslant\ell_\varphi}\tau_x h\Big)\Big] - \frac{1}{2\ell}\mathcal{D}_\ell(\mathbb{P}_1^\star;h)\Big\}.$$

Since φ is quadratic, we can restrict the supremum in the class of quadratic functions h with support contained in Λ_ℓ (the proof of that statement is detailed in Proposition A.3). We can also restrict the supremum to functions h such that $\mathcal{D}_\ell(\mathbb{P}_1^\star;h)\leqslant \mathrm{C}\ell$, as a standard consequence of Proposition 2.2 (namely, there is some constant $\mathrm{C}(\varphi)$ such that the right hand side is non-positive when $\mathcal{D}_\ell(\mathbb{P}_1^\star;h) > \mathrm{C}(\varphi)\ell$). Next, we want to replace the sums over Λ_{ℓ_φ} with the same sums over Λ_ℓ (recall that $\ell_\varphi = \ell - s_\varphi - 1 \leqslant \ell$). For that purpose, we denote

$$\zeta_0^\ell(h) = \nabla_0\Big(\frac{1}{2\ell}\sum_{x=-\ell}^{\ell}\tau_x h\Big), \qquad \zeta_1^\ell(h) = \nabla_{0,1}\Big(\frac{1}{2\ell}\sum_{x=-\ell+1}^{\ell}\tau_x h\Big). \tag{29}$$

First of all, from the Cauchy-Schwarz inequality, we have

$$\mathbb{E}_1^\star\Big[\frac{\gamma}{2}\Big(\zeta_0^\ell(h)\Big)^2 + \frac{\lambda}{2}\Big(\zeta_1^\ell(h)\Big)^2\Big] \leqslant \frac{1}{2\ell}\mathcal{D}_\ell(\mathbb{P}_1^\star;h).$$

Then, from elementary computations (similar to the proof of Proposition 2.2), we can write

$$\Big|\mathbb{E}_1^\star\Big[\varphi,\frac{1}{2\ell}\sum_{\ell_\varphi\leqslant x\leqslant \ell}\tau_x h\Big]\Big| \leqslant \frac{1}{2\ell}\mathrm{C}(\varphi,\gamma)\big(\mathcal{D}_\ell(\mathbb{P}_1^\star;h)\big)^{1/2},$$

where $\mathrm{C}(\varphi,\gamma)$ is a constant which depends only on φ and γ. These last two inequalities give the upper bound

$$(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi,\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi\Big]$$
$$\leqslant \sup_{\substack{h\in\mathcal{Q}\\ \mathcal{D}_\ell(\mathbb{P}_1^\star;h)\leqslant \mathrm{C}\ell}}\Big\{2\mathbb{E}_1^\star\Big[\mathrm{F}\,\zeta_0^\ell(h) + \mathrm{G}\,\zeta_1^\ell(h)\Big] - \mathbb{E}_1^\star\Big[\frac{\gamma}{2}\Big(\zeta_0^\ell(h)\Big)^2 + \frac{\lambda}{2}\Big(\zeta_1^\ell(h)\Big)^2\Big]\Big\} + \frac{\mathrm{C}}{\sqrt{\ell}}, \tag{30}$$

from some constant $\mathrm{C}>0$. From now on, we denote generically by C a positive constant that does not depend on ℓ, but may depend on φ (and γ), and may change from line to line. The conclusion is now based on the following lemma:

Lemma 4.4. *Assume that $h\in\mathcal{Q}$ with support in Λ_ℓ. From Definition 2.2, it reads as*

$$h(\mathbf{m},\omega) = \sum_{\substack{i,j=-\ell\\ i\neq j}}^{\ell}\psi_{i,j}(\mathbf{m})\omega_i\omega_j + \sum_{i=-\ell}^{\ell-1}\psi_{i,i}(\mathbf{m})(\omega_{i+1}^2-\omega_i^2).$$

Then there exists $a_\ell(\mathbf{m})$ and $\mathcal{R}_\ell(\mathbf{m},\omega)$ such that,

$$\zeta_0^\ell(h) = \nabla_0\big(\Gamma_{h/(2\ell)}\big) \tag{31}$$
$$\zeta_1^\ell(h) = \nabla_{0,1}\big(\Gamma_{h/(2\ell)}\big) + a_\ell(\mathbf{m})(\omega_0^2-\omega_1^2) + \mathcal{R}_\ell(\mathbf{m},\omega). \tag{32}$$

Moreover, if $\mathcal{D}_\ell(\mathbb{P}_1^\star; h) \leqslant C\ell$, then

$$\mathbb{E}_1^\star\Big[\big(\mathcal{R}_\ell(\mathbf{m},\omega)\big)^2\Big] \leqslant \frac{C}{\gamma\ell}. \tag{33}$$

Proof of Lemma 4.4. The proof of this lemma is rather straightforward, we merely sketch it. First, given the shape of the function h, elementary computations yield that (31) and (32) hold with

$$a_\ell(\mathbf{m}) = \frac{1}{2\ell}\big(\psi_{\ell-1,\ell-1}(\tau_{-\ell}\mathbf{m}) + \psi_{-\ell,-\ell}(\tau_{\ell+1}\mathbf{m})\big),$$

$$\mathcal{R}_\ell(\mathbf{m},\omega) = \frac{1}{\ell}\Bigg(-\sum_{j=-\ell}^{\ell-1}\psi_{\ell,j}(\tau_{-\ell}\mathbf{m})\,\omega_{j-\ell} + \sum_{j=-\ell+1}^{\ell}\psi_{-\ell,j}(\tau_{\ell+1}\mathbf{m})\,\omega_{j+\ell+1}\Bigg)(\omega_1-\omega_0).$$

Then, we straightforwardly obtain, by translation invariance of $\mathbb{P}_1^\star$,

$$\mathbb{E}_1^\star\Big[\big(\mathcal{R}_\ell(\mathbf{m},\omega)\big)^2\Big] \leqslant \frac{C_1}{\ell^2}\Bigg(\mathbb{E}_1^\star\Bigg[\sum_{j=-\ell}^{\ell-1}\big(\psi_{\ell,j}(\mathbf{m})\big)^2\Bigg] + \mathbb{E}_1^\star\Bigg[\sum_{j=-\ell+1}^{\ell}\big(\psi_{-\ell,j}(\mathbf{m})\big)^2\Bigg]\Bigg),$$

where $C_1 = 4\mathbb{E}_1^\star[\omega_{-1}^2(\omega_1-\omega_0)^2] = 8\mathbb{E}_1^\star[\omega_{-1}^2\omega_0^2] = 8$. Furthermore, since both parts of the Dirichlet form given in (7) are non-negative, in particular, we have

$$\begin{aligned}
\mathcal{D}_\ell(\mathbb{P}_1^\star; h) &\geqslant \frac{\gamma}{2}\sum_{x\in\Lambda_\ell}\mathbb{E}_1^\star\Big[\big(h(\mathbf{m},\omega^x) - h(\mathbf{m},\omega)\big)^2\Big] \\
&\geqslant \frac{\gamma}{2}\mathbb{E}_1^\star\Big[\big(h(\mathbf{m},\omega^\ell) - h(\mathbf{m},\omega)\big)^2\Big] + \frac{\gamma}{2}\mathbb{E}_1^\star\Big[\big(h(\mathbf{m},\omega^{-\ell}) - h(\mathbf{m},\omega)\big)^2\Big] \\
&= \frac{\gamma}{2}\mathbb{E}_1^\star\Bigg[16\omega_\ell^2\Bigg(\sum_{j=-\ell}^{\ell-1}\psi_{\ell,j}(\mathbf{m})\omega_j\Bigg)^2\Bigg] + \frac{\gamma}{2}\mathbb{E}_1^\star\Bigg[16\omega_{-\ell}^2\Bigg(\sum_{j=-\ell+1}^{\ell}\psi_{-\ell,j}(\mathbf{m})\omega_j\Bigg)^2\Bigg] \\
&= 8\gamma\mathbb{E}_1^\star\Bigg[\sum_{j=-\ell}^{\ell-1}\big(\psi_{\ell,j}(\mathbf{m})\big)^2\Bigg] + 8\gamma\mathbb{E}_1^\star\Bigg[\sum_{j=-\ell+1}^{\ell}\big(\psi_{-\ell,j}(\mathbf{m})\big)^2\Bigg].
\end{aligned}$$

The previous two bounds finally yield

$$\mathbb{E}_1^\star\Big[\big(\mathcal{R}_\ell(\mathbf{m},\omega)\big)^2\Big] \leqslant \frac{1}{\gamma\ell^2}\mathcal{D}_\ell(\mathbb{P}_1^\star; h),$$

which proves (33). □

Lemma 4.4 above permits to bound the limit as $\ell\to\infty$ of (30) by

$$\begin{aligned}
\sup_{\substack{f\in\mathcal{Q}\\ a:\Omega_{\mathcal{D}}\to\mathbb{R}}}\Bigg\{&2\mathbb{E}_1^\star\Big[F\,\nabla_0\Gamma_f + G\big(a(\mathbf{m})(\omega_0^2-\omega_1^2) + \nabla_{0,1}\Gamma_f\big)\Big] \\
&- \mathbb{E}_1^\star\Big[\frac{\gamma}{2}\big(\nabla_0\Gamma_f\big)^2 + \frac{\lambda}{2}\big(a(\mathbf{m})(\omega_0^2-\omega_1^2) + \nabla_{0,1}\Gamma_f\big)^2\Big]\Bigg\} =: \sup_{\substack{f\in\mathcal{Q}\\ a:\Omega_{\mathcal{D}}\to\mathbb{R}}}\mathfrak{H}(\varphi,a,f),
\end{aligned} \tag{34}$$

where we denote by $\mathfrak{H}(\varphi,a,f)$ the quantity inside brackets. To conclude we want to restrict the supremum on *real numbers a* which do not depend on the disorder. This is done in a similar way as in [11, Lemma 7.7]. To that aim, for any positive ε, fix $a_\varepsilon(\mathbf{m})$ such that

$$\sup_{f\in\mathcal{Q}}\mathfrak{H}(\varphi,a_\varepsilon,f) \geqslant \sup_{\substack{f\in\mathcal{Q}\\ a:\Omega_{\mathcal{D}}\to\mathbb{R}}}\mathfrak{H}(\varphi,a,f) - \varepsilon, \tag{35}$$

and shorten $\widetilde{a}_\varepsilon(\mathbf{m}) := a_\varepsilon(\mathbf{m}) - \mathbb{E}[a_\varepsilon]$. Let us define, for any $x \in \mathbb{Z}$, the function $b_x \in \mathbf{L}^2(\mathbb{P})$ given by

$$b_x(\mathbf{m}) = \begin{cases} \displaystyle\sum_{k=0}^{x-1} \tau_k \widetilde{a}_\varepsilon(\mathbf{m}) & \text{if } x \geqslant 1, \\ \displaystyle\sum_{k=x}^{-1} -\tau_k \widetilde{a}_\varepsilon(\mathbf{m}) & \text{if } x \leqslant -1. \end{cases} \qquad \text{and} \qquad b_0(\mathbf{m}) = 0,$$

which is defined in such a way that for any $x \in \mathbb{Z}$, $b_{x+1}(\mathbf{m}) - b_x(\mathbf{m}) = \tau_x \widetilde{a}_\varepsilon(\mathbf{m})$. For any $n \in \mathbb{N}$, let us introduce the quadratic function

$$g_n(\mathbf{m}, \omega) = -\sum_{x \in \Lambda_n} b_x(\mathbf{m}) \omega_x^2.$$

One can easily check that, for any $z \in \mathbb{Z}$ such that $\{z, z+1\} \subset \Lambda_n$,

$$\nabla_{z,z+1}(g_n) = \tau_z \widetilde{a}_\varepsilon(\mathbf{m})(\omega_{z+1}^2 - \omega_z^2), \qquad \text{and} \qquad \nabla_0(g_n) = 0.$$

Therefore, letting $\widetilde{f}_n := f + g_n/(2n)$ which still belongs to $\mathcal{Q}$, we get

$$\begin{aligned} &\nabla_0 \Gamma_f = \nabla_0 \Gamma_{\widetilde{f}_n} \\ &\nabla_{0,1}\Gamma_f + a_\varepsilon(\mathbf{m})(\omega_0^2 - \omega_1^2) = \nabla_{0,1}\Gamma_{\widetilde{f}_n} + \mathbb{E}[a_\varepsilon](\omega_0^2 - \omega_1^2) + \mathfrak{R}_n(\mathbf{m}, \omega), \end{aligned}$$

where

$$\mathfrak{R}_n(\mathbf{m}, \omega) = \widetilde{a}_\varepsilon(\mathbf{m})(\omega_0^2 - \omega_1^2) - \frac{1}{2n} \nabla_{0,1} \Gamma_{g_n}.$$

We are now going to estimate the $\mathbf{L}^2(\mathbb{P}_1^\star)$-norm of $\mathfrak{R}_n$, as follows: basic computations show that

$$\begin{aligned} \frac{1}{2n} \nabla_{0,1} \Gamma_{g_n} = \widetilde{a}_\varepsilon(\mathbf{m})(\omega_0^2 - \omega_1^2) &+ \frac{1}{2n} \tau_{-n}\Big(b_n(\mathbf{m})(\omega_n^2 - \omega_{n+1}^2)\Big) \\ &+ \frac{1}{2n} \tau_{n+1}\Big(b_{-n-1}(\mathbf{m})(\omega_{-n-1}^2 - \omega_{-n}^2)\Big). \end{aligned}$$

Hence, from the Cauchy-Schwarz inequality and translation invariance of $\mathbb{P}$, it is enough to show that

$$\frac{\mathbb{E}\big[b_n^2\big]}{n^2} = \frac{1}{n^2} \mathbb{E}\bigg[\bigg(\sum_{k=0}^{n-1} \tau_k \widetilde{a}_\varepsilon(\mathbf{m})\bigg)^2\bigg] \xrightarrow[n\to\infty]{} 0, \qquad \text{and} \qquad \frac{\mathbb{E}\big[b_{-n-1}^2\big]}{n^2} \xrightarrow[n\to\infty]{} 0. \tag{36}$$

These convergences are standard consequences of the translation invariance of $\mathbb{P}$: more precisely, let us fix a positive integer p and introduce for any $x \in \mathbb{Z}$ the conditional expectation

$$\widetilde{a}_x^{(\varepsilon,p)} = \mathbb{E}\Big[\tau_x \widetilde{a}_\varepsilon(\mathbf{m}) \;\Big|\; m_y \; ; \; y \in \Lambda_p(x)\Big].$$

From our assumptions, note that $\widetilde{a}_x^{(\varepsilon,p)} = \tau_x \widetilde{a}_0^{(\varepsilon,p)}$ and $\mathbb{E}\big[\widetilde{a}_x^{(\varepsilon,p)}\big] = 0$. As a result,

$$\begin{aligned} \frac{1}{n^2}\mathbb{E}\bigg[\bigg(\sum_{k=0}^{n-1} \tau_k \widetilde{a}_\varepsilon(\mathbf{m})\bigg)^2\bigg] &\leqslant \frac{2}{n^2}\mathbb{E}\bigg[\bigg(\sum_{k=0}^{n-1}\Big\{\tau_k \widetilde{a}_\varepsilon(\mathbf{m}) - \widetilde{a}_k^{(\varepsilon,p)}\Big\}\bigg)^2\bigg] + \frac{2}{n^2}\mathbb{E}\bigg[\bigg(\sum_{k=0}^{n-1} \widetilde{a}_k^{(\varepsilon,p)}\bigg)^2\bigg] \\ &\leqslant 2\mathbb{E}\bigg[\Big\{\widetilde{a}_\varepsilon(\mathbf{m}) - \widetilde{a}_0^{(\varepsilon,p)}\Big\}^2\bigg] + \frac{C(\varepsilon,p)}{n}. \end{aligned}$$

The last inequality comes from the fact that $\sum \widetilde{a}_k^{(\varepsilon,p)}$ is a sum of identically distributed variables (because of the translation invariance of $\mathbb{P}$), for which we have a good control of the variance. Letting now, in the bound above, $n \to \infty$, and then $p \to \infty$, we obtain that (36) holds, thus (35) rewrites

$$\sup_{f \in \mathcal{Q}} \mathfrak{H}(\varphi, \mathbb{E}[a_\varepsilon], f) \geqslant \sup_{\substack{f \in \mathcal{Q} \\ a:\Omega_{\mathcal{D}} \to \mathbb{R}}} \mathfrak{H}(\varphi, a, f) - \varepsilon.$$

Since this holds for any $\varepsilon > 0$, we finally obtain as wanted that

$$\sup_{\substack{f\in\mathcal{Q}\\ a\in\mathbb{R}}} \mathfrak{H}(\varphi,a,f) = \sup_{\substack{f\in\mathcal{Q}\\ a:\Omega_{\mathcal{D}}\to\mathbb{R}}} \mathfrak{H}(\varphi,a,f),$$

and therefore

$$(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi,\ \sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi\Big] \leqslant \sup_{\substack{g\in\mathcal{Q}\\ a\in\mathbb{R}}}\Big\{2\mathbb{E}_1^\star\Big[\mathrm{F}\,\nabla_0\Gamma_g + \mathrm{G}\,\big(a(\omega_0^2-\omega_1^2)+\nabla_{0,1}\Gamma_g\big)\Big]$$
$$-\frac{\gamma}{2}\mathbb{E}_1^\star\Big[(\nabla_0\Gamma_g)^2\Big] - \frac{\lambda}{2}\mathbb{E}_1^\star\Big[\big(a(\omega_0^2-\omega_1^2)+\nabla_{0,1}\Gamma_g\big)^2\Big]\Big\}.$$

Lemma 4.3 follows, after recalling (28).

We now turn to the upper bound.

Lemma 4.5. *Under the assumptions of Theorem 4.2,*

$$\limsup_{\ell\to\infty}(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi,\ \sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi\Big] \geqslant \|\!|\varphi|\!\|_1^2.$$

Proof. We define, for $f\in\mathcal{Q}$, define $\ell_f = \ell - s_f - 1$ and

$$\mathrm{J}_\ell := \sum_{y,y+1\in\Lambda_\ell}\tau_y j_{0,1}^{\mathrm{S}}, \qquad \mathrm{H}_\ell^f = \sum_{|y|\leqslant\ell_f}\mathcal{S}(\tau_y f).$$

The following limits hold:

$$\lim_{\ell\to\infty}(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi,\ \mathrm{J}_\ell\Big] = -\ll\varphi\gg_{1,\star\star}, \tag{37}$$

$$\lim_{\ell\to\infty}(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi,\ \mathrm{H}_\ell^f\Big] = -\ll\varphi,f\gg_{1,\star},$$

$$\lim_{\ell\to\infty}(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\Big(a\mathrm{J}_\ell+\mathrm{H}_\ell^f\Big),\Big(a\mathrm{J}_\ell+\mathrm{H}_\ell^f\Big)\Big] =$$
$$\frac{\lambda}{2}\mathbb{E}_1^\star\Big[\big(a(\omega_0^2-\omega_1^2)+\nabla_{0,1}\Gamma_f\big)^2\Big] + \frac{\gamma}{2}\mathbb{E}_1^\star\Big[(\nabla_0\Gamma_f)^2\Big].$$

We only prove (37), the other relations can be obtained in a similar way. As previously, we assume for the sake of simplicity that $\varphi = \nabla_0(\mathrm{F}) + \nabla_{0,1}(\mathrm{G})$. One can easily check the elementary identity

$$\mathcal{S}_{\Lambda_\ell}\Big(\sum_{x\in\Lambda_\ell}x\,\omega_x^2\Big) = \mathrm{J}_\ell(\omega). \tag{38}$$

Therefore,

$$\begin{aligned}(2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant\ell_\varphi}\tau_x\varphi,\mathrm{J}_\ell\Big] &= -(2\ell)^{-1}\sum_{y\in\Lambda_\ell}\sum_{|x|\leqslant\ell_\varphi} y\ \mathbb{E}_1^\star\big[\varphi\,\omega_{y-x}^2\big]\\ &= -(2\ell)^{-1}\sum_{y\in\Lambda_\ell}\sum_{|x|\leqslant\ell_\varphi} y\ \mathbb{E}_1^\star\big[\mathrm{G}\,\nabla_{0,1}(\omega_{y-x}^2)\big]\\ &= -(2\ell)^{-1}\sum_{|x|\leqslant\ell_\varphi} x\ \mathbb{E}_1^\star\big[\mathrm{G}\,\nabla_{0,1}(\omega_0^2)\big] + (x+1)\mathbb{E}_1^\star\big[\mathrm{G}\,\nabla_{0,1}(\omega_1^2)\big]\\ &= -(2\ell)^{-1}(2\ell_\varphi+1)\ \mathbb{E}_1^\star\big[\mathrm{G}(\omega_0^2-\omega_1^2)\big] \xrightarrow[\ell\to\infty]{} -\ll\varphi\gg_{1,\star\star}.\end{aligned}$$

The last limit comes from Proposition 2.5 and the fact that $\ell_\varphi = \ell - s_\varphi - 1$. We also have used the translation invariance of $\mathbb{P}_1^\star$. Then, we use the variational formula (11), chosing

$$h = (\mathcal{S}_{\Lambda_\ell})^{-1}(a\mathrm{J}_\ell + \mathrm{H}_\ell^f) = a\sum_{y\in\Lambda_\ell} y\omega_y^2 + \sum_{|y|\leqslant \ell_f} \tau_y f,$$

we obtain:

$$\begin{aligned}
&\liminf_{\ell\to\infty} (2\ell)^{-1}\mathbb{E}_1^\star\Big[\big(-\mathcal{S}_{\Lambda_\ell}\big)^{-1}\sum_{|x|\leqslant \ell_\varphi}\tau_x\varphi, \sum_{|x|\leqslant \ell_\varphi}\tau_x\varphi\Big] \\
&\quad\geqslant \liminf_{\ell\to\infty}(2\ell)^{-1}\Big\{2\mathbb{E}_1^\star\Big[\sum_{|x|\leqslant \ell_\varphi}\tau_x\varphi, a\sum_{y\in\Lambda_\ell} y\omega_y^2 + \sum_{|y|\leqslant \ell_f}\tau_y f\Big] + \mathbb{E}_1^\star\Big[a\sum_{y\in\Lambda_\ell} y\omega_y^2 + \sum_{|y|\leqslant \ell_f}\tau_y f, a\mathrm{J}_\ell + \mathrm{H}_\ell^f\Big]\Big\} \\
&\quad= 2a \ll \varphi \gg_{1,\star\star} + 2 \ll \varphi, f \gg_{1,\star} - \frac{\lambda}{2}\mathbb{E}_1^\star\Big[\big(a(\omega_0^2-\omega_1^2) + \nabla_{0,1}\Gamma_f\big)^2\Big] - \frac{\gamma}{2}\mathbb{E}_1^\star\Big[\big(\nabla_0\Gamma_f\big)^2\Big].
\end{aligned}$$

The result follows after taking the supremum on $f\in\mathcal{Q}$ and $a\in\mathbb{R}$, and recalling (28). □

4.3 Proof of Theorem 3.3

In this paragraph, we start the proof of Theorem 3.3 by using the result given in Theorem 4.2. First, we show how to relate (21) to such variances, as was rapidly sketched in Section 4.1. Recall that we have assumed for convenience $\beta = 1$, but the same argument remains in force for any $\beta > 0$.

PROPOSITION 4.6. *Let $\psi\in\mathcal{C}_0$, with $s_\psi \leqslant \mathrm{N}$. Then*

$$\mathbb{E}^\star_{\mu_1^{\mathrm{N}}}\left[\sup_{0\leqslant t\leqslant \mathrm{T}}\left\{\int_0^t \psi(s)\,\mathrm{d}s\right\}^2\right] \leqslant \frac{24\mathrm{T}}{\mathrm{N}^2}\mathbb{E}_1^\star\big[\psi, (-\mathcal{S}_{\mathrm{N}})^{-1}\psi\big]. \tag{39}$$

This result is proved for example in [18, Section 2, Lemma 2.4], when there is no disorder. The average w.r.t. the disorder can be added (as in the estimate (39)) without any trouble, since $\mathcal{S}_{\mathrm{N}}$ does not depend on **m**. We are going to use this bound for functions of type $\sum_x \mathrm{G}(x/\mathrm{N})\tau_x\varphi$, where φ belongs to $\mathcal{Q}_0$. The main result of this subsection is the following.

THEOREM 4.7. *Let $\varphi\in\mathcal{Q}_0$, and $\mathrm{G}\in\mathrm{C}^2(\mathbb{T})$. Then,*

$$\limsup_{\mathrm{N}\to\infty}\mathbb{E}^\star_{\mu_1^{\mathrm{N}}}\left[\sup_{0\leqslant t\leqslant \mathrm{T}}\left\{\sqrt{\mathrm{N}}\int_0^t\sum_{x\in\mathbb{T}_{\mathrm{N}}}\mathrm{G}\left(\frac{x}{\mathrm{N}}\right)\tau_x\varphi(\mathbf{m},s)\mathrm{d}s\right\}^2\right] \leqslant \mathrm{C}\mathrm{T}|\!|\!|\varphi|\!|\!|_1^2\int_{\mathbb{T}}\mathrm{G}^2(u)\mathrm{d}u. \tag{40}$$

Proof. From Proposition 4.6, the left hand side of (40) is bounded by

$$24\mathrm{T}\,\mathbb{E}_1^\star\Big[\sqrt{\mathrm{N}}\sum_{x\in\mathbb{T}_{\mathrm{N}}}\mathrm{G}\left(\frac{x}{\mathrm{N}}\right)\tau_x\varphi, (-\mathrm{N}^2\mathcal{S}_{\mathrm{N}})^{-1}\Big(\sqrt{\mathrm{N}}\sum_{x\in\mathbb{T}_{\mathrm{N}}}\mathrm{G}\left(\frac{x}{\mathrm{N}}\right)\tau_x\varphi\Big)\Big],$$

which can be written with the variational formula as

$$24\mathrm{T}\sup_{f\in\mathcal{C}}\Big\{\sqrt{\mathrm{N}}\sum_{x\in\mathbb{T}_{\mathrm{N}}}\mathrm{G}\left(\frac{x}{\mathrm{N}}\right)\mathbb{E}_1^\star[f\,\tau_x\varphi] - \mathrm{N}^2\mathcal{D}_{\mathrm{N}}(\mathbb{P}_1^\star;f)\Big\}.$$

Since $\varphi\in\mathcal{Q}_0$, from Proposition A.3 we can restrict the supremum over $f\in\mathcal{Q}$. Proposition 2.2 gives

$$\mathbb{E}_1^\star[f\,\tau_x\varphi] \leqslant \mathrm{C}(\varphi,\gamma)\mathbb{E}_1^\star\Big[\tau_{-x}f, (-\mathcal{S}_{\Lambda_\varphi})(\tau_{-x}f)\Big]^{1/2}$$

and by Cauchy-Schwarz inequality,

$$\sqrt{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\mathbb{E}_1^\star[f\,\tau_x\varphi]\leqslant\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)^2\right)^{1/2} N\,C(\varphi,\gamma)\,\mathbb{E}_1^\star\big[f,(-\mathcal{S}_N)f\big]^{1/2}.$$

The supremum on f can be explicitly computed, and gives the final bound

$$\mathbb{E}^\star_{\mu_1^N}\left[\sup_{0\leqslant t\leqslant T}\left\{\sqrt{N}\int_0^t\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\tau_x\varphi(\mathbf{m},s)\mathrm{d}s\right\}^2\right]\leqslant C'(\varphi,\gamma)T\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)^2\right). \tag{41}$$

We are now going to show that, after sending N to infinity, the constant on the right hand side is proportional to $|||\varphi|||_1^2$. For that purpose, we average on microscopic boxes: for $\ell\ll N$, we denote

$$\overline{\varphi}_\ell=\frac{1}{2\ell_\varphi+1}\sum_{|y|\leqslant\ell_\varphi}\tau_y\varphi,$$

where as before $\ell_\varphi=\ell-s_\varphi-1$. We want to substitute

$$\sqrt{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\tau_x\varphi$$

with

$$\sqrt{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\tau_x\overline{\varphi}_\ell.$$

The error term that appears is estimated by

$$\mathbb{E}^\star_{\mu_1^N}\left[\sup_{0\leqslant t\leqslant T}\left\{\sqrt{N}\int_0^t\sum_{\substack{x,y\in\mathbb{T}_N\\|x-y|\leqslant\ell_\varphi}}\frac{1}{2\ell_\varphi+1}\left(G\left(\frac{x}{N}\right)-G\left(\frac{y}{N}\right)\right)\tau_x\varphi(\mathbf{m},s)\mathrm{d}s\right\}^2\right].$$

Since $G_\ell(x):=G(x/N)-(2\ell_\varphi+1)^{-1}\sum_{|y-x|\leqslant\ell_\varphi}G(y/N)$ is of order ℓ/N, we obtain from (41) that the expression above is bounded by $C(\ell)/N^2$, and therefore vanishes as $N\to\infty$. We are now reduced to estimate

$$\mathbb{E}^\star_{\mu_1^N}\left[\sup_{0\leqslant t\leqslant T}\left\{\sqrt{N}\int_0^t\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\tau_x\overline{\varphi}_\ell(\mathbf{m},s)\mathrm{d}s\right\}^2\right]. \tag{42}$$

Using once again (39) and the variational formula for its right hand side, one obtains straightforwardly, using the translation invariance of $\mathbb{P}_1^\star$, that (42) is bounded by

$$\begin{aligned}
&CT\sup_{g\in\mathcal{Q}}\left\{\sqrt{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\mathbb{E}_1^\star[g\,\tau_x\overline{\varphi}_\ell]-N^2\mathbb{E}_1^\star[g,(-\mathcal{S}_N)\,g]\right\}\\
&\quad\leqslant CT\sup_{g\in\mathcal{Q}}\left\{\sqrt{N}\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\mathbb{E}_1^\star[\tau_{-x}g\,\overline{\varphi}_\ell]-\frac{N^2}{2\ell+1}\sum_{x\in\mathbb{T}_N}\mathbb{E}_1^\star\Big[g,\big(-\mathcal{S}_{\Lambda_\ell(x)}\big)g\Big]\right\}\\
&\quad\leqslant\frac{CT(2\ell+1)}{N}\sum_{x\in\mathbb{T}_N}G^2\left(\frac{x}{N}\right)\sup_{f\in\mathcal{Q}}\left\{\mathbb{E}_1^\star[f\,\overline{\varphi}_\ell]-\mathbb{E}_1^\star[f,(-\mathcal{S}_{\Lambda_\ell})\,f]\right\}\\
&\quad\leqslant\frac{CT(2\ell+1)}{N}\sum_{x\in\mathbb{T}_N}G^2\left(\frac{x}{N}\right)\sup_{f\in\mathcal{Q}_\ell}\left\{\mathbb{E}_1^\star[f\,\overline{\varphi}_\ell]-\mathbb{E}_1^\star[f,(-\mathcal{S}_{\Lambda_\ell})\,f]\right\},
\end{aligned}$$

where in the last inequality we denote by $\mathcal{Q}_\ell$ the set of functions in $\mathcal{Q}$ depending only on the sites in $\Lambda_{\ell-1}$. To obtain the second bound, we split the supremum over x, and let $f:=(2\ell+1)\tau_{-x}g/G(x/N)$, and to obtain the third bound, we used the convexity of the Dirichlet form to replace f by its conditional expectation w.r.t. sites in $\Lambda_{\ell-1}$. Since $\varphi \in \mathcal{Q}_0$, from Corollary 2.3, one straightforwardly obtains, using the polarization identity related to (7) and the elementary inequality $ab \leqslant \frac{1}{4}a^2 + b^2$, that

$$\begin{aligned}
\mathbb{E}_1^\star[f\,\overline{\varphi}_\ell] &= \mathbb{E}_1^\star\big[f(-\mathcal{S}_{\Lambda_\ell})(-\mathcal{S}_{\Lambda_\ell})^{-1}\overline{\varphi}_\ell\big] \\
&= \frac{\gamma}{2}\sum_{x=-\ell}^{\ell}\mathbb{E}_1^\star\big[\nabla_x f, \nabla_x\big((-\mathcal{S}_{\Lambda_\ell})^{-1}\overline{\varphi}_\ell\big)\big] + \frac{\lambda}{2}\sum_{x=-\ell}^{\ell-1}\mathbb{E}_1^\star\big[\nabla_{x,x+1}f, \nabla_{x,x+1}\big((-\mathcal{S}_{\Lambda_\ell})^{-1}\overline{\varphi}_\ell\big)\big] \\
&\leqslant \frac{1}{4}\mathbb{E}_1^\star\big[\overline{\varphi}_\ell, (-\mathcal{S}_{\Lambda_\ell})^{-1}\overline{\varphi}_\ell\big] + \mathbb{E}_1^\star\big[f(-\mathcal{S}_{\Lambda_\ell})f\big].
\end{aligned}$$

We can now plug this bound in the previous estimate, let $\ell \to \infty$ after $\mathrm{N} \to \infty$ and use Theorem 4.2 to finally obtain as wanted

$$\limsup_{\mathrm{N}\to\infty}\mathbb{E}_{\mu_1^{\mathrm{N}}}^\star\left[\sup_{0\leqslant t\leqslant \mathrm{T}}\left\{\sqrt{\mathrm{N}}\int_0^t\sum_{x\in\mathbb{T}_\mathrm{N}}\mathrm{G}\left(\frac{x}{\mathrm{N}}\right)\tau_x\varphi(\mathbf{m},s)\mathrm{d}s\right\}^2\right] \leqslant \mathrm{C}\mathrm{T}\lvert\!\lvert\!\lvert\varphi\rvert\!\rvert\!\rvert_1^2\int_\mathbb{T}\mathrm{G}^2(u)\mathrm{d}u. \tag{43}$$

□

We apply Theorem 4.7 to $\mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H})$, and we get

$$\limsup_{\mathrm{N}\to\infty}\mathbb{E}_{\mu_1^{\mathrm{N}}}^\star\left[\sup_{0\leqslant t\leqslant \mathrm{T}}\left(\mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H})\right)^2\right] \leqslant \mathrm{C}\mathrm{T}\lvert\!\lvert\!\lvert j_{0,1} + \mathrm{D}(\omega_1^2-\omega_0^2) + \mathcal{L}^{\mathbf{m}}f\rvert\!\rvert\!\rvert_1^2\int_\mathbb{T}\mathrm{H}'(u)^2\mathrm{d}u.$$

To conclude the proof of Theorem 3.3, we show in Section 5 that there exists a sequence of local functions $\{f_k\} \in \mathcal{Q}$ such that

$$\lvert\!\lvert\!\lvert j_{0,1} + \mathrm{D}(\omega_1^2-\omega_0^2) + \mathcal{L}^{\mathbf{m}}f_k\rvert\!\rvert\!\rvert_1 \xrightarrow[k\to\infty]{} 0,$$

and Section 6 is devoted to prove the second statement of Theorem 3.3.

5 Hilbert space and projections

We now focus on the semi-norm $\lvert\!\lvert\!\lvert\cdot\rvert\!\rvert\!\rvert_1$ that was introduced in the previous section, see (27). We can easily define from $\lvert\!\lvert\!\lvert\cdot\rvert\!\rvert\!\rvert_1$ a semi inner product on $\mathcal{C}_0$ through polarization, which is denoted by $\ll\cdot,\cdot\gg_1$. Let $\mathcal{N}$ be the kernel of the semi-norm $\lvert\!\lvert\!\lvert\cdot\rvert\!\rvert\!\rvert_1$ on $\mathcal{C}_0$. Then, the completion of $\mathcal{Q}_0|_\mathcal{N}$ denoted by $\mathcal{H}_1$ is a Hilbert space. Let us explain how Varadhan's non-gradient approach is modified. Usually, the Hilbert space on which orthogonal projections are performed is the completion of $\mathcal{C}_0|_\mathcal{N}$, in other words it involves all local functions. Then, the standard procedure aims at proving that each element of that Hilbert space can be approximated by a sequence of functions in the range of the generator plus an additional term which is proportional to the current. Since for our model, all functions of interest are in $\mathcal{Q}$, and since the decomposition of germs of closed form is explicit in the set $\mathcal{Q}$ (recall (31) and (32)), the crucial step to obtain this decomposition is to control the antisymmetric part of the generator by the symmetric one for quadratic functions.

In Subsection 5.1, we show that $\mathcal{H}_1$ is the completion of $\mathcal{S}\mathcal{Q}|_\mathcal{N}+\{j_{0,1}^{\mathrm{S}}\}$. In other words, all elements of $\mathcal{H}_1$ can be approximated by $aj_{0,1}^{\mathrm{S}}+\mathcal{S}g$ for some $a\in\mathbb{R}$ and $g \in \mathcal{Q}$. This is quite natural since the symmetric part of the generator preserves the degree of polynomial functions. Moreover, the two subspaces $\{j_{0,1}^{\mathrm{S}}\}$ and $\overline{\mathcal{S}\mathcal{Q}}|_\mathcal{N}$ are orthogonal, and we denote their sum by

$$\overline{\mathcal{S}\mathcal{Q}}|_\mathcal{N}\oplus^\perp\{j_{0,1}^{\mathrm{S}}\}.$$

Nevertheless, this decomposition is not satisfactory, because we want the fluctuating term to be on the form $\mathcal{L}^{\mathbf{m}}(f_k)$, and not $\mathcal{S}(f_k)$. In order to make this replacement, we need to prove the weak sector condition, that gives a control of $\|\mathcal{A}^{\mathbf{m}} g\|_1$ by $\|\mathcal{S}g\|_1$, when g is a quadratic function. The argument is explained is Subsection 5.2 and 5.3, and the weak sector condition is proved in Appendix B. The only trouble is that this new decomposition is no longer orthogonal, so that we can not directly express the diffusion coefficient as a variational formula, like (49). This problem is solved in Section 6.

5.1 Decomposition according to the symmetric part

We begin this subsection with a table of calculus, very useful in the sequel. Recall that $\ll \cdot,\cdot \gg_1$ is obtained by polarization from the norm $\|\cdot\|_1$ defined in Definition 4.1, and also that $\ll \cdot \gg_{1,\star}$ and $\ll \cdot \gg_{1,\star\star}$ have been defined in (12).

Proposition 5.1. *For any $\varphi \in \mathcal{Q}_0$ and $g \in \mathcal{Q}$ (which implies $\mathcal{S}g \in \mathcal{Q}_0$ from Proposition 2.1),*

$$\begin{aligned} \ll \varphi, \mathcal{S}g \gg_1 &= - \ll \varphi, g \gg_{1,\star} \\ \ll \varphi, j^{S}_{0,1} \gg_1 &= - \ll \varphi \gg_{1,\star\star} \\ \ll j^{S}_{0,1}, \mathcal{S}g \gg_1 &= 0 \end{aligned}$$

and then

$$\begin{aligned} \|j^{S}_{0,1}\|_1^2 &= - \ll j^{S}_{0,1} \gg_{1,\star\star} = \lambda\chi(1) \\ \|\mathcal{S}g\|_1^2 &= \frac{\lambda}{2}\mathbb{E}_1^\star\left[(\nabla_{0,1}\Gamma_g)^2\right] + \frac{\gamma}{2}\mathbb{E}_1^\star\left[(\nabla_0\Gamma_g)^2\right] \end{aligned}$$

Proof. These identities are direct consequences of Theorem 4.2. The second one uses (38). The third one uses Remark 2.1. □

Corollary 5.2. *For all $a \in \mathbb{R}$ and $g \in \mathcal{Q}$,*

$$\|a j^{S}_{0,1} + \mathcal{S}g\|_1^2 = a^2\lambda\chi(1) + \frac{\lambda}{2}\mathbb{E}_1^\star\left[(\nabla_{0,1}\Gamma_g)^2\right] + \frac{\gamma}{2}\mathbb{E}_1^\star\left[(\nabla_0\Gamma_g)^2\right].$$

In particular, the variational formula for $\|\varphi\|_1$, $\varphi \in \mathcal{Q}_0$, writes

$$\|\varphi\|_1^2 = \frac{1}{\lambda\chi(1)} \ll \varphi, j^{S}_{0,1} \gg_1^2 + \sup_{g\in\mathcal{Q}}\left\{2 \ll \varphi, (-\mathcal{S})g \gg_1 - \|\mathcal{S}g\|_1^2\right\}. \tag{44}$$

Proposition 5.3. *We denote by $\mathcal{S}\mathcal{Q}$ the space $\{\mathcal{S}g\ ;\ g \in \mathcal{Q}\}$. Then,*

$$\mathcal{H}_1 = \overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}} \oplus^\perp \{j^{S}_{0,1}\}.$$

Proof. We divide the proof into two steps.

(a) The space is well generated – The inclusion $\overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}} + \{j^{S}_{0,1}\} \subset \mathcal{H}_1$ is obvious (and follows from Proposition 2.1). Moreover, from the variational formula (44) we know that: if $h \in \mathcal{H}_1$ satisfies $\ll h, j^{S}_{0,1} \gg_1 = 0$ and $\ll h, \mathcal{S}g \gg_1 = 0$ for all $g \in \mathcal{Q}$, then $\|h\|_1 = 0$.

(b) The sum is orthogonal – This follows directly from the previous proposition and from the fact that: $\ll j^{S}_{0,1}, \mathcal{S}g \gg_1 = 0$ for all $g \in \mathcal{Q}$. □

5.2 Replacement of $\mathcal{S}$ with $\mathcal{L}$

In this subsection, we prove identities which mix the antisymmetric and the symmetric part of the generator, which will be used to get the *weak* sector condition (Proposition 5.7).

Lemma 5.4. *For all $g, h \in \mathcal{Q}$,*

$$\ll \mathcal{S}g, \mathcal{A}^{\mathbf{m}}h \gg_1 = - \ll \mathcal{A}^{\mathbf{m}}g, \mathcal{S}h \gg_1 .$$

Proof. This easily follows from the first identity of Proposition 5.1 and from the invariance by translation of the measure $\mathbb{P}_1^\star$:

$$\begin{aligned} \ll \mathcal{S}g, \mathcal{A}^{\mathbf{m}}h \gg_1 &= - \ll g, \mathcal{A}^{\mathbf{m}}h \gg_{1,\star} = - \sum_{x\in\mathbb{Z}} \mathbb{E}_1^\star\big[\tau_x g, \mathcal{A}^{\mathbf{m}}h\big] = \sum_{x\in\mathbb{Z}} \mathbb{E}_1^\star\big[\mathcal{A}^{\mathbf{m}}(\tau_x g), h\big] \\ &= \sum_{x\in\mathbb{Z}} \mathbb{E}_1^\star\big[\tau_x(\mathcal{A}^{\mathbf{m}}g), h\big] = \sum_{x\in\mathbb{Z}} \mathbb{E}_1^\star\big[\mathcal{A}^{\mathbf{m}}g, \tau_{-x}h\big] = \sum_{x\in\mathbb{Z}} \mathbb{E}_1^\star\big[\mathcal{A}^{\mathbf{m}}g, \tau_x h\big] = - \ll \mathcal{A}^{\mathbf{m}}g, \mathcal{S}h \gg_1 . \end{aligned}$$

□

Lemma 5.5. *For all $g \in \mathcal{Q}$,*

$$\ll \mathcal{S}g, j_{0,1}^{\mathrm{A}} \gg_1 = - \ll \mathcal{A}^{\mathbf{m}}g, j_{0,1}^{\mathrm{S}} \gg_1 .$$

Proof. From Proposition 5.1,

$$\begin{aligned} \ll \mathcal{S}g, j_{0,1}^{\mathrm{A}} \gg_1 &= - \ll g, j_{0,1}^{\mathrm{A}} \gg_{1,\star} = - \sum_{x\in\mathbb{Z}} \mathbb{E}_1^\star\big[\tau_x g, j_{0,1}^{\mathrm{A}}\big] = - \sum_{x\in\mathbb{Z}} \mathbb{E}_1^\star\big[g, j_{x,x+1}^{\mathrm{A}}\big] \\ &= - \sum_{x\in\mathbb{Z}} x\mathbb{E}_1^\star\big[g, j_{x-1,x}^{\mathrm{A}} - j_{x,x+1}^{\mathrm{A}}\big] = - \sum_{x\in\mathbb{Z}} x\mathbb{E}_1^\star\big[g, \mathcal{A}^{\mathbf{m}}(\omega_x^2)\big] \\ &= \sum_{x\in\mathbb{Z}} x\mathbb{E}_1^\star\big[\mathcal{A}^{\mathbf{m}}g, \omega_x^2\big] = \ll \mathcal{A}^{\mathbf{m}}g \gg_{1,\star\star} = - \ll \mathcal{A}^{\mathbf{m}}g, j_{0,1}^{\mathrm{S}} \gg_1 . \end{aligned}$$

□

These two lemmas together with the second identity of Proposition 5.1 (and the fact that $\ll j_{0,1}^{\mathrm{A}} \gg_{1,\star\star} = 0$) imply the following:

Corollary 5.6. *For all $a \in \mathbb{R}$, $g \in \mathcal{Q}$,*

$$\ll a j_{0,1}^{\mathrm{S}} + \mathcal{S}g, a j_{0,1}^{\mathrm{A}} + \mathcal{A}^{\mathbf{m}}g \gg_1 = 0.$$

We now state the main result of this subsection.

Proposition 5.7 (Weak sector condition). *(i) There exist two constants $\mathrm{C}_0 := \mathrm{C}(\gamma,\lambda)$ and $\mathrm{C}_1 := \mathrm{C}(\gamma,\lambda)$ such that the following inequalities hold for all $f, g \in \mathcal{Q}$:*

$$\big| \ll \mathcal{A}^{\mathbf{m}}f, \mathcal{S}g \gg_1 \big| \leqslant \mathrm{C}_0 \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathcal{S}f \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1 \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathcal{S}g \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1 . \tag{45}$$

$$\big| \ll \mathcal{A}^{\mathbf{m}}f, \mathcal{S}g \gg_1 \big| \leqslant \mathrm{C}_1 \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathcal{S}f \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1^2 + \frac{1}{2} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathcal{S}g \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1^2 . \tag{46}$$

(ii) There exists a positive constant C such that, for all $g \in \mathcal{Q}$,

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathcal{A}^{\mathbf{m}}g \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1 \leqslant \mathrm{C} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathcal{S}g \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1 .$$

Proof. The proof is technical because made of explicit computations for quadratic functions. For that reason, we report it to Appendix B. □

5.3 Decomposition of the Hilbert space

We deduce from the previous two subsections the expected decomposition of $\mathcal{H}_1$.

PROPOSITION 5.8. *We denote by $\mathcal{L}^{\mathbf{m}}\mathcal{Q}$ the space $\{\mathcal{L}^{\mathbf{m}}g\ ;\ g \in \mathcal{Q}\}$. Then,*

$$\mathcal{H}_1 = \overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}}|_{\mathcal{N}} \oplus \{j^{\mathrm{S}}_{0,1}\}.$$

Proof. We first prove that $\mathcal{H}_1$ can be written as the sum of the two subspaces. Then, we show that the sum is direct.

(a) The space is well generated – The inclusion $\overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}}|_{\mathcal{N}} + \{j^{\mathrm{S}}_{0,1}\} \subset \mathcal{H}_1$ follows from Proposition 2.1. To prove the converse inclusion, let $h \in \mathcal{H}_1$ so that $\ll h, j^{\mathrm{S}}_{0,1} \gg_1 = 0$ and $\ll h, \mathcal{L}^{\mathbf{m}}g \gg_1 = 0$ for all $g \in \mathcal{Q}$. From Proposition 5.3, h can be written as

$$h = \lim_{k\to\infty} \mathcal{S}g_k$$

for some sequence $\{g_k\} \in \mathcal{Q}$. More precisely, since $\ll \mathcal{S}g_k, \mathcal{A}^{\mathbf{m}}g_k \gg_1 = 0$ by Lemma 5.4,

$$|||h|||_1^2 = \lim_{k\to\infty} \ll \mathcal{S}g_k, \mathcal{S}g_k \gg_1 = \lim_{k\to\infty} \ll \mathcal{S}g_k, \mathcal{L}^{\mathbf{m}}g_k \gg_1 .$$

Moreover, we also have by assumption that $\ll h, \mathcal{L}^{\mathbf{m}}g_k \gg_1 = 0$ for all k, and from Proposition 5.7,

$$\sup_{k\in\mathbb{N}} |||\mathcal{L}^{\mathbf{m}}g_k|||_1 \leqslant (\mathrm{C}+1) \sup_{k\in\mathbb{N}} |||\mathcal{S}g_k|||_1 =: \mathrm{C}_h$$

is finite. Therefore,

$$|||h|||_1^2 = \lim_{k\to\infty} \ll \mathcal{S}g_k, \mathcal{L}^{\mathbf{m}}g_k \gg_1 = \lim_{k\to\infty} \ll \mathcal{S}g_k - h, \mathcal{L}^{\mathbf{m}}g_k \gg_1 \leqslant \lim_{k\to\infty} \mathrm{C}_h |||\mathcal{S}g_k - h|||_1 = 0.$$

(b) The sum is direct – Let $\{g_k\} \in \mathcal{Q}$ be a sequence such that, for some $a \in \mathbb{R}$,

$$\lim_{k\to\infty} \mathcal{L}^{\mathbf{m}}g_k = a j^{\mathrm{S}}_{0,1} \quad \text{in } \mathcal{H}_1,$$

By a similar argument,

$$\limsup_{k\to\infty} \ll \mathcal{S}g_k, \mathcal{S}g_k \gg_1 = \limsup_{k\to\infty} \ll \mathcal{L}^{\mathbf{m}}g_k, \mathcal{S}g_k \gg_1 = \limsup_{k\to\infty} \ll \mathcal{L}^{\mathbf{m}}g_k - a j^{\mathrm{S}}_{0,1}, \mathcal{S}g_k \gg_1 = 0,$$

where the last equality comes from the fact that $\ll j^{\mathrm{S}}_{0,1}, \mathcal{S}g_k \gg_1 = 0$ for all k. On the other hand, by Proposition 5.7, $|||\mathcal{L}^{\mathbf{m}}g_k|||_1 \leqslant (\mathrm{C}+1)|||\mathcal{S}g_k|||_1 \to 0$. Then, $a = 0$. This concludes the proof. □

Recall that $j^{\mathrm{S}}_{0,1}(\mathbf{m}, \omega) = \lambda(\omega_0^2 - \omega_1^2)$. We have obtained the following result.

THEOREM 5.9. *For every $g \in \mathcal{Q}_0$, there exists a unique constant $a \in \mathbb{R}$, such that*

$$g + a(\omega_1^2 - \omega_0^2) \in \overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}} \quad \textit{in } \mathcal{H}_1. \tag{47}$$

In particular, this theorem states that there exists a unique number $\widetilde{\mathrm{D}}$, and a sequence of cylinder functions $\{f_k\} \in \mathcal{Q}$ such that

$$|||j_{0,1} + \widetilde{\mathrm{D}}(\omega_1^2 - \omega_0^2) + \mathcal{L}^{\mathbf{m}}f_k|||_1 \xrightarrow[k\to\infty]{} 0. \tag{48}$$

For any quadratic function $f \in \mathcal{Q}$, $|||f|||_\beta = \beta^{-2}|||f|||_1$, therefore in particular, this convergence also holds with the same constant $\widetilde{\mathrm{D}}$ and the same sequence f_k if we replace the semi-norm $|||\cdot|||_1$ with $|||\cdot|||_\beta$ for any $\beta > 0$ (as a consequence of a standard change of variables argument). This concludes the first statement of Theorem 3.3. We prove the second statement (22) in Proposition 6.5, Section 6.

6 On the diffusion coefficient

The main goal of this section is to express the diffusion coefficient by various variational formulas. We also prove the second statement of Theorem 3.3. First, recall the we defined the coefficient D in Definition 2.4 as

$$\mathrm{D} = \lambda + \frac{1}{\chi(1)} \inf_{f\in\mathcal{Q}} \sup_{g\in\mathcal{Q}} \Big\{ \ll f, -\mathcal{S}f \gg_{1,\star} + 2 \ll j^{\mathrm{A}}_{0,1} - \mathcal{A}^{\mathbf{m}} f, g \gg_{1,\star} - \ll g, -\mathcal{S}g \gg_{1,\star} \Big\}. \tag{49}$$

From Theorem 5.9, there exists a unique $\widetilde{\mathrm{D}} \in \mathbb{R}$ such that

$$j_{0,1} + \widetilde{\mathrm{D}}(\omega_1^2 - \omega_0^2) \in \overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}} \quad \text{ in } \mathcal{H}_1.$$

We are going to obtain variational formulas for $\widetilde{\mathrm{D}}$, and prove that $\widetilde{\mathrm{D}} = \mathrm{D}$, by following the argument in [23]. We first rewrite the decomposition of the Hilbert space given in Proposition 5.8, by replacing $j^{\mathrm{S}}_{0,1}$ with $j_{0,1}$. This new statement is based on Corollary 5.6, which gives an orthogonality relation. The second step is to find an other orthogonal decomposition (see (50) below), which will enable us to prove the variational formula (49) for D. Hereafter, we denote $\mathcal{L}^{\mathbf{m},\star} := \mathcal{S} - \mathcal{A}^{\mathbf{m}}$ and $j^{\star}_{0,1} := j^{\mathrm{S}}_{0,1} - j^{\mathrm{A}}_{0,1}$.

Lemma 6.1. *The following decompositions hold*

$$\mathcal{H}_1 = \overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}}|_{\mathcal{N}} \oplus \{j_{0,1}\} = \overline{\mathcal{L}^{\mathbf{m},\star}\mathcal{Q}}|_{\mathcal{N}} \oplus \{j^{\star}_{0,1}\}.$$

Proof. We only sketch the proof of the first decomposition, since it is done in [23]. Let us recall from Proposition 5.8 that $\overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}}$ has a complementary subspace in $\mathcal{H}_1$ which is one-dimensional. Therefore, it is sufficient to prove that $\mathcal{H}_1$ is generated by $\overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}}$ and the total current. Let $h \in \mathcal{H}_1$ such that $\ll h, j_{0,1} \gg_1 = 0$ and $\ll h, \mathcal{L}^{\mathbf{m}} g \gg_1 = 0$ for all $g \in \mathcal{Q}$. By Proposition 5.3, h can be written as

$$h = \lim_{k\to\infty} \mathcal{S}g_k + a j^{\mathrm{S}}_{0,1}$$

for some sequence $\{g_k\} \in \mathcal{Q}$, and $a \in \mathbb{R}$, and from Corollary 5.6,

$$|||h|||_1^2 = \lim_{k\to\infty} \ll a j^{\mathrm{S}}_{0,1} + \mathcal{S}g_k, a j_{0,1} + \mathcal{L}^{\mathbf{m}} g_k \gg_1 .$$

Moreover, from Proposition 5.7 and the standard inequality $|||\varphi + \psi|||_1^2 \leqslant 2|||\varphi|||_1^2 + 2|||\psi|||_1^2$, we have

$$\sup_{k\in\mathbb{N}} |||a j_{0,1} + \mathcal{L}^{\mathbf{m}} g_k|||_1^2 \leqslant 2a^2 |||j_{0,1}|||_1^2 + 2(\mathrm{C}+1) \sup_{k\in\mathbb{N}} |||\mathcal{S}g_k|||_1^2 =: \mathrm{C}_h$$

is finite. Therefore,

$$\begin{aligned} |||h|||_1^2 &= \lim_{k\to\infty} \ll a j^{\mathrm{S}}_{0,1} + \mathcal{S}g_k - h, a j_{0,1} + \mathcal{L}^{\mathbf{m}} g_k \gg_1 \\ &\leqslant \mathrm{C}_h \limsup_{k\to\infty} \left|\left|\left| a j^{\mathrm{S}}_{0,1} + \mathcal{S}g_k - h \right|\right|\right|_1 = 0. \end{aligned}$$

The same arguments apply to the second decomposition. □

We define bounded linear operators $\mathrm{T}, \mathrm{T}^{\star} : \mathcal{H}_1 \to \mathcal{H}_1$ as

$$\begin{aligned} \mathrm{T}(a j_{0,1} + \mathcal{L}^{\mathbf{m}} f) &:= a j^{\mathrm{S}}_{0,1} + \mathcal{S}f, \\ \mathrm{T}^{\star}(a j^{\star}_{0,1} + \mathcal{L}^{\mathbf{m},\star} f) &:= a j^{\mathrm{S}}_{0,1} + \mathcal{S}f. \end{aligned}$$

From the following identity (which is a direct consequence of Corollary 5.6)

$$\left\|\left|a j_{0,1}+\mathcal{L}^{\mathbf{m}} f\right\|\right|_1^2=\left\|\left|a j_{0,1}^{\star}+\mathcal{L}^{\mathbf{m}, \star} f\right\|\right|_1^2=\left\|\left|a j_{0,1}^{\mathrm{S}}+\mathcal{S} f\right\|\right|_1^2+\left\|\left|a j_{0,1}^{\mathrm{A}}+\mathcal{A}^{\mathbf{m}} f\right\|\right|_1^2,$$

we can easily see that $\mathrm{T}^{\star}$ is the adjoint operator of T and we also have the relations

$$\begin{aligned} & \ll \mathrm{T} j_{0,1}^{\mathrm{S}}, j_{0,1}^{\star} \gg_1=\ll \mathrm{T}^{\star} j_{0,1}^{\mathrm{S}}, j_{0,1} \gg_1=\lambda \chi(1) \\ & \ll \mathrm{T} j_{0,1}^{\mathrm{S}}, \mathcal{L}^{\mathbf{m}, \star} f \gg_1=\ll \mathrm{T}^{\star} j_{0,1}^{\mathrm{S}}, \mathcal{L}^{\mathbf{m}} f \gg_1=0, \text { for all } f \in \mathcal{Q} . \end{aligned}$$

In particular,

$$\mathcal{H}_1=\left.\overline{\mathcal{L}^{\mathbf{m}, \star} \mathcal{Q}}\right|_{\mathcal{N}} \oplus^{\perp}\left\{\mathrm{T} j_{0,1}^{\mathrm{S}}\right\} \tag{50}$$

and there exists a unique number Q such that

$$j_{0,1}^{\star}-\mathrm{QT} j_{0,1}^{\mathrm{S}} \in \overline{\mathcal{L}^{\mathbf{m}, \star} \mathcal{Q}} \quad \text { in } \mathcal{H}_1 .$$

We are going to show that $\widetilde{\mathrm{D}}=\lambda \mathrm{Q}$.

Lemma 6.2.

$$\mathrm{Q}=\frac{\lambda \chi(1)}{\left\|\left|\mathrm{T} j_{0,1}^{\mathrm{S}}\right\|\right|_1^2}=\frac{1}{\lambda \chi(1)} \inf _{f \in \mathcal{Q}}\left\|\left|j_{0,1}^{\star}-\mathcal{L}^{\mathbf{m}, \star} f\right\|\right|_1^2 . \tag{51}$$

Proof. The first identity follows from the fact that

$$\ll \mathrm{T} j_{0,1}^{\mathrm{S}}, j_{0,1}^{\star}-\mathrm{QT} j_{0,1}^{\mathrm{S}} \gg_1=\lambda \chi(1)-\mathrm{Q}\left\|\left|\mathrm{T} j_{0,1}^{\mathrm{S}}\right\|\right|_1^2=0 .$$

The second identity is straightforwardly obtained from the first identity, together with

$$\inf _{f \in \mathcal{Q}}\left\|\left|j_{0,1}^{\star}-\mathrm{QT} j_{0,1}^{\mathrm{S}}-\mathcal{L}^{\mathbf{m}, \star} f\right\|\right|_1^2=0, \tag{52}$$

which holds by construction of Q. □

Thanks to Corollary 5.6, for any $g \in \mathcal{Q}$, $\mathrm{T}g$ and $g-\mathrm{T}g$ are orthogonal, and therefore $\ll \mathrm{T} g, g \gg_1=\ll \mathrm{T} g, \mathrm{T} g \gg_1$ for all $g \in \mathcal{H}_1$. In particular, $j_{0,1}^{\mathrm{S}}-\mathrm{T} j_{0,1}^{\mathrm{S}}$ is orthogonal to $\mathrm{T} j_{0,1}^{\mathrm{S}}$, thus

$$j_{0,1}^{\mathrm{S}}-\mathrm{T} j_{0,1}^{\mathrm{S}} \in \overline{\mathcal{L}^{\mathbf{m}, \star} \mathcal{Q}} .$$

We can then obtain the following variational formula for $\left\|\left|\mathrm{T} j_{0,1}^{\mathrm{S}}\right\|\right|_1$.

Proposition 6.3.

$$\left\|\left|\mathrm{T} j_{0,1}^{\mathrm{S}}\right\|\right|_1^2=\inf _{f \in \mathcal{Q}}\left\|\left|j_{0,1}^{\mathrm{S}}-\mathcal{L}^{\mathbf{m}, \star} f\right\|\right|_1^2 . \tag{53}$$

Proof. With a similar argument (as in the proof of the previous proposition), we have

$$\inf _{f \in \mathcal{Q}}\left\|\left|j_{0,1}^{\mathrm{S}}-\mathrm{T} j_{0,1}^{\mathrm{S}}-\mathcal{L}^{\mathbf{m}, \star} f\right\|\right|_1=0,$$

and

$$\inf _{f \in \mathcal{Q}}\left\|\left|j_{0,1}^{\mathrm{S}}-\mathrm{T} j_{0,1}^{\mathrm{S}}-\mathcal{L}^{\mathbf{m}, \star} f\right\|\right|_1^2=\inf _{f \in \mathcal{Q}}\left\|\left|j_{0,1}^{\mathrm{S}}-\mathcal{L}^{\mathbf{m}, \star} f\right\|\right|_1^2-\left\|\left|\mathrm{T} j_{0,1}^{\mathrm{S}}\right\|\right|_1^2,$$

where we used the fact that $j_{0,1}^{\mathrm{S}}-\mathrm{T} j_{0,1}^{\mathrm{S}}$ and $\mathcal{L}^{\mathbf{m}, \star} f$ are both orthogonal to $\mathrm{T} j_{0,1}^{\mathrm{S}}$, which concludes the proof. □

We are now ready to derive variational formulas for $\widetilde{\mathrm{D}}$:

THEOREM 6.4.

$$\widetilde{\mathrm{D}} = \frac{1}{\chi(1)} \inf_{f\in\mathcal{Q}} \left\|\left| j^{\star}_{0,1} - \mathcal{L}^{\mathbf{m},\star} f \right\|\right|_1^2 = \frac{\chi(1)}{4\inf_{f\in\mathcal{Q}} \left\|\left| j^{\mathrm{S}}_{0,1} - \mathcal{L}^{\mathbf{m},\star} f \right\|\right|_1^2}. \tag{54}$$

Proof. By construction, $j_{0,1} - (\widetilde{\mathrm{D}}/\lambda) j^{\mathrm{S}}_{0,1} \in \overline{\mathcal{L}^{\mathbf{m}}\mathcal{Q}}$ and therefore

$$\ll j_{0,1} - \frac{\widetilde{\mathrm{D}}}{\lambda} j^{\mathrm{S}}_{0,1}, \mathrm{T}^{\star} j^{\mathrm{S}}_{0,1} \gg_1 = \lambda\chi(1) - \frac{\widetilde{\mathrm{D}}}{\lambda} \left\|\left| \mathrm{T} j^{\mathrm{S}}_{0,1} \right\|\right|_1^2 = 0. \tag{55}$$

As a result, we obtain as wanted that, $\widetilde{\mathrm{D}} = \lambda \mathrm{Q}$, and the variational formula for $\widetilde{\mathrm{D}}$ can be deduced from the one for Q. □

REMARK 6.1. We can rewrite the variational formula (54) for $\widetilde{\mathrm{D}}$ as:

$$\begin{aligned}
\widetilde{\mathrm{D}} &= \frac{1}{\chi(1)} \inf_{f\in\mathcal{Q}} \left\{ \left\|\left| j^{\mathrm{S}}_{0,1} \right\|\right|_1^2 + \left\|\left| \mathcal{S} f \right\|\right|_1^2 + \left\|\left| j^{\mathrm{A}}_{0,1} - \mathcal{A}^{\mathbf{m}} f \right\|\right|_1^2 \right\} \\
&= \lambda + \frac{1}{\chi(1)} \inf_{f\in\mathcal{Q}} \left\{ \left\|\left| \mathcal{S} f \right\|\right|_1^2 + \left\|\left| j^{\mathrm{A}}_{0,1} - \mathcal{A}^{\mathbf{m}} f \right\|\right|_1^2 \right\} && (56) \\
&= \lambda + \frac{1}{\chi(1)} \inf_{f\in\mathcal{Q}} \sup_{g\in\mathcal{Q}} \left\{ \left\|\left| \mathcal{S} f \right\|\right|_1^2 - 2 \ll j^{\mathrm{A}}_{0,1} - \mathcal{A}^{\mathbf{m}} f, \mathcal{S} g \gg_1 - \left\|\left| \mathcal{S} g \right\|\right|_1^2 \right\} \\
&= \lambda + \frac{1}{\chi(1)} \inf_{f\in\mathcal{Q}} \sup_{g\in\mathcal{Q}} \left\{ \ll f, -\mathcal{S} f \gg_{1,\star} + 2 \ll j^{\mathrm{A}}_{0,1} - \mathcal{A}^{\mathbf{m}} f, g \gg_{1,\star} - \ll g, -\mathcal{S} g \gg_{1,\star} \right\} && (57) \\
&= \mathrm{D}, && (58)
\end{aligned}$$

by definition of the diffusion coefficient, see (49). To establish the third identity, we used (44) to restrict the infimum in (56), to functions f satisfying $\ll j^{\mathrm{A}}_{0,1} - \mathcal{A}^{\mathbf{m}} f, j^{\mathrm{S}}_{0,1} \gg_1 = 0$.

We are now in position to prove the remaining statement of Theorem 3.3:

PROPOSITION 6.5. *For any sequence $\{f_k\} \in \mathcal{Q}$ such that*

$$\lim_{k\to\infty} \left\|\left| j_{0,1} + \mathrm{D}(\omega_1^2 - \omega_0^2) + \mathcal{L}^{\mathbf{m}} f_k \right\|\right|_1 = 0$$

we have

$$\lim_{k\to\infty} \mathbb{E}^{\star}_1 \left[\lambda \left(\nabla_{0,1}(\omega_0^2 - \Gamma_{f_k}) \right)^2 + \gamma \left(\nabla_0(\Gamma_{f_k}) \right)^2 \right] = 2\mathrm{D}\chi(1).$$

Proof. By assumption,

$$\lim_{k\to\infty} \left\|\left| \mathrm{T}\left(j_{0,1} + \mathrm{D}(\omega_1^2 - \omega_0^2) + \mathcal{L}^{\mathbf{m}} f_k \right) \right\|\right|_1 = 0$$

and therefore

$$\lim_{k\to\infty} \left\|\left| j^{\mathrm{S}}_{0,1} + \mathcal{S} f_k \right\|\right|_1^2 = \mathrm{D}^2 \left\|\left| \mathrm{T}(\omega_1^2 - \omega_0^2) \right\|\right|_1^2.$$

Then, the result follows from

$$\mathrm{D} = \lambda \mathrm{Q} = \frac{\chi(1)}{\left\|\left| \mathrm{T}(\omega_1^2 - \omega_0^2) \right\|\right|_1^2}$$

and Corollary 5.2, which yields

$$\left\|\left| j^{\mathrm{S}}_{0,1} + \mathcal{S} f_k \right\|\right|_1^2 = \frac{\lambda}{2} \mathbb{E}^{\star}_1 \left[\left(\omega_1^2 - \omega_0^2 - \nabla_{0,1}(\Gamma_{f_k}) \right)^2 \right] + \frac{\gamma}{2} \mathbb{E}^{\star}_1 \left[\left(\nabla_0(\Gamma_{f_k}) \right)^2 \right]. \tag{59}$$

□

7 Green-Kubo formulas

In this section, we first prove the convergence of the infinite volume Green-Kubo formula, then we rigorously show that it is equivalent to the diffusion coefficient given by Varadhan's approach. For the sake of clarity, in the following we simplify notations, and we denote $\ll \cdot \gg_{1,\star}$ by $\ll \cdot \gg_{\star}$.

7.1 Convergence of Green-Kubo formula

Linear response theory predicts that the diffusion coefficient is given by the Green-Kubo formula. In [3, Section 3] its homogenized infinite volume version is given by:

$$\overline{\kappa}(z) = \lambda + \frac{1}{2}\int_0^{+\infty} \mathrm{d}t\ e^{-zt} \sum_{x\in\mathbb{Z}} \mathbb{E}^{\star}_{\mu_1^{\mathrm{N}}}\Big[j^{\mathrm{A}}_{0,1}(\mathbf{m},t), \tau_x j^{\mathrm{A}}_{0,1}(\mathbf{m},0)\Big]. \tag{60}$$

That formula can be guessed from the better-known finite volume Green-Kubo formula thanks to the ergodicity property of the disorder measure $\mathbb{P}$. We denote by $\mathrm{L}(z)$ the second term of the right hand side of (60), that is

$$\mathrm{L}(z) := \frac{1}{2}\int_0^{+\infty} \mathrm{d}t e^{-zt} \sum_{x\in\mathbb{Z}} \mathbb{E}^{\star}_{\mu_1^{\mathrm{N}}}\Big[j^{\mathrm{A}}_{0,1}(\mathbf{m},t), \tau_x j^{\mathrm{A}}_{0,1}(\mathbf{m},0)\Big].$$

We also denote by $\mathbf{L}^2_\star$ the Hilbert space generated by the elements of $\mathcal{C}$ (recall Definition 2.1) and the inner product $\ll \cdot \gg_\star$. We define $h_z := h_z(\mathbf{m},\omega)$ as the solution to the resolvent equation in $\mathbf{L}^2_\star$

$$(z - \mathcal{L}^{\mathbf{m}})h_z = j^{\mathrm{A}}_{0,1}. \tag{61}$$

Hille-Yosida Theorem (see Proposition 2.1 in [10] for instance) implies that the Laplace transform $\mathrm{L}(z)$ is well defined, is smooth on $(0,+\infty)$, and such that

$$\overline{\kappa}(z) = \lambda + \mathrm{L}(z) = \lambda + \frac{1}{2} \ll j^{\mathrm{A}}_{0,1}, h_z \gg_\star . \tag{62}$$

Since the generator $\mathcal{L}^{\mathbf{m}}$ conserves the degree of homogeneous polynomial functions, the solution to the resolvent equation is on the form

$$h_z(\mathbf{m},\omega) = \sum_{x\in\mathbb{Z}} \varphi_z(\mathbf{m},x,x)(\omega^2_{x+1} - \omega^2_x) + \sum_{\substack{x,y\in\mathbb{Z}\\ x\neq y}} \varphi_z(\mathbf{m},x,y)\omega_x\omega_y,$$

where, for all $\mathbf{m} \in \Omega_{\mathcal{D}}$, the function $\varphi_z(\mathbf{m},\cdot,\cdot) : \mathbb{Z}^2 \to \mathbb{R}$ is square-summable and symmetric. In the Hilbert space $\mathbf{L}^2_\star$, there exist equations involving the symmetric part $\mathcal{S}$ that can be explicitly solved:

Lemma 7.1. *There exists $f \in \mathbf{L}^2_\star$ such that $\mathcal{S}f = j^{\mathrm{A}}_{0,1}$ in $\mathbf{L}^2_\star$.*

Proof. We look at the solutions f to $\mathcal{S}f = j^{\mathrm{A}}_{0,1}$ on the form

$$f(\mathbf{m},\omega) = \sum_{\substack{x\in\mathbb{Z}\\ k\geqslant 1}} \varphi_k(\mathbf{m},x)\omega_x\omega_{x+k},$$

such that, for all $\mathbf{m} \in \Omega_{\mathcal{D}}$,

$$\sum_{\substack{x\in\mathbb{Z}\\ k\geqslant 1}} |\varphi_k(\mathbf{m},x)|^2 < +\infty. \tag{63}$$

To simplify notations, let us erase the dependence of $\mathbf{m}$ for a while, and keep it in mind. Then, the sequence $\{\varphi_k(x); x \in \mathbb{Z}, k \geqslant 1\}$ has to be solution to

$$\begin{cases} -\lambda\big(\varphi_{k+1}(x)+\varphi_{k+1}(x-1)\big)+4(\lambda+\gamma)\varphi_k(x)-\lambda\big(\varphi_{k-1}(x)+\varphi_{k-1}(x-1)\big)=0, \text{ for } k\geqslant 2, x\in\mathbb{Z}, \\ (\lambda+2\gamma)\varphi_1(x)-\lambda\big(\varphi_2(x)+\varphi_2(x-1)\big)=\dfrac{\delta_0(x)}{\sqrt{m_0m_1}}, \quad \text{for } x\in\mathbb{Z}. \end{cases} \tag{64}$$

We introduce the Fourier transform $\widehat{\varphi}_k$ defined on $\mathbb{T}$ as follows:

$$\widehat{\varphi}_k(\xi) := \sum_{x\in\mathbb{Z}} \varphi_k(x)e^{2i\pi x\xi}, \qquad \xi\in\mathbb{T}.$$

From (63), this is well defined, and the inverse Fourier transform together with the Plancherel-Parseval relation imply:

$$\begin{cases} \varphi_k(x) = \displaystyle\int_{\mathbb{T}} \widehat{\varphi}_k(\xi)e^{-2i\pi x\xi}\mathrm{d}\xi \\ \displaystyle\sum_{\substack{x\in\mathbb{Z}\\ k\geqslant 1}} |\varphi_k(x)|^2 = \sum_{k\geqslant 1}\int_{\mathbb{T}} |\widehat{\varphi}_k(\xi)|^2\mathrm{d}\xi. \end{cases}$$

The system of equations (64) rewrites as

$$\begin{cases} -\lambda\big(e^{2i\pi\xi}+1\big)\widehat{\varphi}_{k+1}(\xi)+4(\lambda+\gamma)\widehat{\varphi}_k(\xi)-\lambda\big(e^{-2i\pi\xi}+1\big)\widehat{\varphi}_{k-1}(\xi)=0, \quad \text{for } k\geqslant 2, \xi\in\mathbb{T}, \\ (\lambda+2\gamma)\widehat{\varphi}_1(\xi)-\lambda\big(e^{2i\pi\xi}+1\big)\widehat{\varphi}_2(\xi)=\dfrac{1}{\sqrt{m_0m_1}}, \quad \text{for } \xi\in\mathbb{T}. \end{cases} \tag{65}$$

Therefore, for any $\xi\in\mathbb{T}$ fixed, $\xi\neq 1/2$, the sequence $\{\widehat{\varphi}_k(\xi)\}_{k\geqslant 1}$ is solution to the second order linear recurrence relation:

$$\widehat{\varphi}_{k+1}(\xi)-\frac{2\alpha e^{-i\pi\xi}}{\cos(\pi\xi)}\widehat{\varphi}_k(\xi)+e^{-2i\pi\xi}\widehat{\varphi}_{k-1}(\xi)=0, \quad \text{for } k\geqslant 2, \tag{66}$$

where $\alpha := (\lambda+\gamma)/\lambda$, with the two conditions:

$$\begin{cases} (\lambda+2\gamma)\widehat{\varphi}_1(\xi)-\lambda\big(e^{2i\pi\xi}+1\big)\widehat{\varphi}_2(\xi)=\theta(\mathbf{m}), \quad \text{for } \xi\in\mathbb{T}, \\ \displaystyle\sum_{k\geqslant 1}\int_{\mathbb{T}} |\widehat{\varphi}_k(\xi)|^2\mathrm{d}\xi < +\infty, \end{cases}$$

where $\theta(\mathbf{m}) := 1/\sqrt{m_0m_1}$. This system is explicitely solvable, and one can easily check that the following function is solution:

$$\widehat{\varphi}_k(\xi) = \frac{\theta(\mathbf{m})}{\gamma r(\xi)+\lambda\big(1+e^{-2i\pi\xi}\big)}\big(r(\xi)\big)^{k-1},$$

where

$$r(\xi) := \frac{\alpha e^{-i\pi\xi}}{\cos(\pi\xi)}\left(1-\sqrt{1-\alpha^{-2}\cos^2(\pi\xi)}\right).$$

Note that $r(\cdot)$ is continuous on $\mathbb{T}$ (from a direct Taylor expansion), and $\varphi_k(x)$ can then be written as the inverse Fourier transform of $\widehat{\varphi}_k(\xi)$. □

We are now able to prove the existence of the Green-Kubo formula:

Theorem 7.2. *The following limit*

$$\overline{\mathrm{D}} := \lim_{\substack{z\to 0\\ z>0}} \overline{\kappa}(z) \tag{67}$$

exists, and is finite.

Proof. We investigate the existence of the limit

$$\lim_{\substack{z\to 0\\ z>0}} \ll j^{A}_{0,1}, (z-\mathcal{L}^{\mathbf{m}})^{-1} j^{A}_{0,1} \gg_\star = 2 \lim_{\substack{z\to 0\\ z>0}} \mathrm{L}(z). \tag{68}$$

With the notations above, we have to prove that $\ll h_z, j^{A}_{0,1} \gg_\star$ converges as z goes to 0, and that the limit is finite and non-negative. Then, from (62) it will follow that $\overline{\mathrm{D}} \geqslant \lambda > 0$ and $\overline{\mathrm{D}}$ is positive. We denote by $\|\cdot\|_{\mathbb{1}\star}$ the semi-norm corresponding to the symmetric part of the generator:

$$\|f\|^2_{\mathbb{1}\star} = \ll f, (-\mathcal{S})f \gg_\star$$

and $\mathbb{H}_{\mathbb{1}\star}$ is the Hilbert space obtained by the completion of $\mathcal{C}$ w.r.t. that semi-norm. The corresponding dual norm is defined as

$$\|f\|^2_{-\mathbb{1}\star} := \sup_{g\in\mathcal{C}} \big\{ 2 \ll f, g \gg_\star - \|g\|^2_{\mathbb{1}\star} \big\}. \tag{69}$$

We denote by $\mathbb{H}_{-\mathbb{1}\star}$ the Hilbert space obtained by the completion of $\mathcal{C}$ w.r.t. that norm. We already know from the previous sections that $\mathcal{Q}_0 \subset \mathbb{H}_{-\mathbb{1}\star}$ (and we recover the result of Lemma 7.1, namely $j^{A}_{0,1} \in \mathbb{H}_{-\mathbb{1}\star}$).

We are going to prove the existence of the Green-Kubo formula by using some arguments given in [18, Section 2.6]. For the reader's convenience, we recall here the main steps, and refer to [18] for the technical details of the proof. First, we take the inner product $\ll \cdot, \cdot \gg_\star$ of (61) and h_z to obtain

$$z \ll h_z, h_z \gg_\star + \|h_z\|^2_{\mathbb{1}\star} = \ll h_z, j^{A}_{0,1} \gg_\star . \tag{70}$$

Since $j^{A}_{0,1} \in \mathbb{H}_{-\mathbb{1}\star}$, the Cauchy-Schwarz inequality for the scalar product $\ll \cdot \gg_\star$ gives

$$z \ll h_z, h_z \gg_\star + \|h_z\|^2_{\mathbb{1}\star} \leqslant \|h_z\|_{\mathbb{1}\star} \, \|j^{A}_{0,1}\|_{-\mathbb{1}\star}$$

and we obtain that

$$\|h_z\|_{\mathbb{1}\star} \leqslant \|j^{A}_{0,1}\|_{-\mathbb{1}\star}.$$

The family $\{h_z\}_{z>0}$ is therefore bounded in $\mathbb{H}_{\mathbb{1}\star}$, and one can extract a weakly converging subsequence in $\mathbb{H}_{\mathbb{1}\star}$. We continue to denote this subsequence by $\{h_z\}$ and we denote by h_0 the limit. We also have

$$z \ll h_z, h_z \gg_\star \leqslant \|j^{A}_{0,1}\|^2_{-\mathbb{1}\star},$$

and then $\{zh_z\}$ strongly converges to 0 in $\mathbf{L}^2_\star$. We now invoke the weak sector condition given in Proposition 5.7: there exists $\mathrm{C}_0 > 0$ such that, for any homogeneous polynomials of degree two $f, g \in \mathbb{H}_{\mathbb{1}\star}$,

$$\big| \ll f, \mathcal{L}^{\mathbf{m}} g \gg_\star \big| \leqslant (\mathrm{C}_0 + 1) \|f\|_{\mathbb{1}\star} \, \|g\|_{\mathbb{1}\star} . \tag{71}$$

Indeed, this is a consequence of (45), since

$$\big| \ll f, \mathcal{A}^{\mathbf{m}} g \gg_\star \big| = \big| \ll \mathcal{S}f, \mathcal{A}^{\mathbf{m}} g \gg_1 \big| \leqslant \mathrm{C}_0 \, \big\|\!\big|\mathcal{S}f\big|\!\big\|_1 \, \big\|\!\big|\mathcal{S}g\big|\!\big\|_1 = \mathrm{C}_0 \|f\|_{\mathbb{1}\star} \, \|g\|_{\mathbb{1}\star},$$

and we also have from the Cauchy-Schwarz inquality,

$$\big| \ll f, \mathcal{S} g \gg_\star \big| \leqslant \|f\|_{\mathbb{1}\star} \, \|g\|_{\mathbb{1}\star} .$$

The estimate given in (71) applied to $g = h_z$ yields

$$\|\mathcal{L}^{\mathbf{m}} h_z\|_{-\mathbb{1}\star} = \sup_{f\in\mathcal{C}} \big\{ 2 \ll f, \mathcal{L}^{\mathbf{m}} h_z \gg_\star - \|f\|^2_{\mathbb{1}\star} \big\} \leqslant (\mathrm{C}_0+1) \|h_z\|^2_{\mathbb{1}\star} \leqslant (\mathrm{C}_0+1) \|j^{A}_{0,1}\|^2_{-\mathbb{1}\star}. \tag{72}$$

From (61) and (72) we deduce that

$$\sup_{z>0} \|zh_z\|_{-\mathbb{1}\star} < \infty.$$

Let us know refer to [18, Section 2.6, Lemma 2.16]: the condition (72) is sufficient to prove that

- the sequence $\{(-\mathcal{L}^{\mathbf{m}})h_z\}$ weakly converges to $j^{\mathrm{A}}_{0,1}$ in $\mathbb{H}_{-\mathbb{1}\star}$;
- the following identity holds

$$\ll h_0, (-\mathcal{L}^{\mathbf{m}})h_0 \gg_\star = \ll h_0, (-\mathcal{S})h_0 \gg_\star = \ll h_0, j^{\mathrm{A}}_{0,1} \gg_\star ; \tag{73}$$

- the sequence $\{h_z\}$ strongly converges to h_0 in $\mathbb{H}_{\mathbb{1}\star}$, and the limit is unique.

We have proved the first part: the limit (68) exists. To obtain its finiteness, we are going to give an upper bound, using the following variational formula:

$$\ll j^{\mathrm{A}}_{0,1}, (z-\mathcal{L}^{\mathbf{m}})^{-1} j^{\mathrm{A}}_{0,1} \gg_\star = \sup_{f\in\mathcal{C}} \left\{ 2 \ll f, j^{\mathrm{A}}_{0,1} \gg_\star - \|f\|^2_{1,z} - \|\mathcal{A}^{\mathbf{m}} f\|^2_{-1,z} \right\},$$

where the two norms $\|\cdot\|_{\pm 1,z}$ are defined by

$$\|f\|^2_{\pm 1,z} = \ll f, (z-\mathcal{S})^{\pm 1} f \gg_\star .$$

For the upper bound, we neglect the term coming from the antisymmetric part $\mathcal{A}^{\mathbf{m}} f$, which gives

$$\ll j^{\mathrm{A}}_{0,1}, (z-\mathcal{L}^{\mathbf{m}})^{-1} j^{\mathrm{A}}_{0,1} \gg_\star \leqslant \ll j^{\mathrm{A}}_{0,1}, (z-\mathcal{S})^{-1} j^{\mathrm{A}}_{0,1} \gg_\star .$$

In the right hand side we can also neglect the part coming from the exchange symmetric part $\mathcal{S}^{\mathrm{exch}}$, and remind that $\mathcal{S}^{\mathrm{flip}}(j^{\mathrm{A}}_{0,1}) = -2j^{\mathrm{A}}_{0,1}$. This gives an explicit finite upper bound. Then, we have from (73) that

$$\lim_{z\to 0} \ll j^{\mathrm{A}}_{0,1}, (z-\mathcal{L}^{\mathbf{m}})^{-1} j^{\mathrm{A}}_{0,1} \gg_\star = \ll j^{\mathrm{A}}_{0,1}, h_0 \gg_\star = \ll h_0, (-\mathcal{S})h_0 \gg_\star \geqslant 0,$$

and the positiveness is proved. □

7.2 Equivalence of the definitions

In this subsection we rigorously prove the equality between the variational formula for the diffusion coefficient and the Green-Kubo formula.

THEOREM 7.3. *For every $\lambda > 0$ and $\gamma > 0$,*

$$\overline{\mathrm{D}} = \lambda + \frac{1}{2} \lim_{\substack{z\to 0\\ z>0}} \ll j^{\mathrm{A}}_{0,1}, (z-\mathcal{L}^{\mathbf{m}})^{-1} j^{\mathrm{A}}_{0,1} \gg_\star$$

coincides with the coefficient $\widetilde{\mathrm{D}} = \mathrm{D}$ defined in Theorem 6.4.

Proof. From Subsection 6, we know that the diffusion coefficient can be written different ways. For instance, since $\chi(1) = 2$, we have

$$\mathrm{D} = \frac{2}{|||\mathrm{T}(\omega_1^2 - \omega_0^2)|||_1^2}.$$

By definition of D, there exists a sequence $\{f_\varepsilon\}_{\varepsilon>0}$ of functions in $\mathcal{Q}$ such that

$$g_\varepsilon := j^\star_{0,1} + \mathrm{D}(\omega_1^2 - \omega_0^2) + \mathcal{L}^{\mathbf{m},\star} f_\varepsilon$$

satisfies $|||g_\varepsilon|||_1 \to 0$ as ε goes to 0. Observe that $g_\varepsilon \in \mathcal{Q}_0 \subset \mathbb{H}_{-\mathbb{1}\star}$ from Proposition 2.1. By substitution in the equality above, we get

$$\mathrm{D}^{-1} = \frac{1}{2\mathrm{D}^2} \ll g_\varepsilon - j^\star_{0,1} - \mathcal{L}^{\mathbf{m},\star} f_\varepsilon, \mathrm{T}^\star(g_\varepsilon - j^\star_{0,1} - \mathcal{L}^{\mathbf{m},\star} f_\varepsilon) \gg_1$$

recalling that $\ll \mathrm{T}g, \mathrm{T}g \gg_1 = \ll g, \mathrm{T}^\star g \gg_1$ for all $g \in \mathcal{H}_1$. Therefore,

$$\begin{aligned} \mathrm{D} &= \frac{1}{2} \ll g_\varepsilon - j^\star_{0,1} - \mathcal{L}^{\mathbf{m},\star} f_\varepsilon, g_\varepsilon - j^{\mathrm{S}}_{0,1} - \mathcal{S} f_\varepsilon \gg_1 \\ &= \frac{1}{2} \ll j^\star_{0,1} + \mathcal{L}^{\mathbf{m},\star} f_\varepsilon, j^{\mathrm{S}}_{0,1} + \mathcal{S} f_\varepsilon \gg_1 + \mathrm{R}_\varepsilon \end{aligned}$$

where R_ε is bounded by $\mathrm{C}|||g_\varepsilon|||_1^2$, and then vanishes as ε goes to 0. Finally, from Proposition 5.1, we can write

$$\mathrm{D} = \lambda + \frac{1}{2} \lim_{\varepsilon \to 0} \ll f_\varepsilon, (-\mathcal{S}) f_\varepsilon \gg_\star ,$$

and we know that the limit above exists, which implies that $\big| \ll f_\varepsilon, (-\mathcal{S}) f_\varepsilon \gg_\star \big|$ is uniformly bounded in ε by a constant $\mathrm{C} > 0$. The problem is now reduced to prove that

$$\lim_{\varepsilon \to 0} \ll f_\varepsilon, (-\mathcal{S}) f_\varepsilon \gg_\star = \lim_{\substack{z \to 0 \\ z > 0}} \ll j^{\mathrm{A}}_{0,1}, (z - \mathcal{L}^{\mathbf{m}})^{-1} j^{\mathrm{A}}_{0,1} \gg_\star . \tag{74}$$

For every $z > 0$ and $\varepsilon > 0$, we have by definition above and (61),

$$j^{\mathrm{A}}_{0,1} = z h_z - \mathcal{L}^{\mathbf{m}} h_z \tag{75}$$

$$j^\star_{0,1} = g_\varepsilon - \mathrm{D}(\omega_1^2 - \omega_0^2) - \mathcal{L}^{\mathbf{m},\star} f_\varepsilon. \tag{76}$$

First, we take the inner product $\ll \cdot, \cdot \gg_\star$ of (76) with f_ε (recall that $\ll j^{\mathrm{S}}_{0,1}, f_\varepsilon \gg_\star = - \ll j^{\mathrm{S}}_{0,1}, \mathcal{S} f_\varepsilon \gg_1 = 0$), to get

$$\ll j^{\mathrm{A}}_{0,1}, f_\varepsilon \gg_\star = - \ll f_\varepsilon, g_\varepsilon \gg_\star - \ll f_\varepsilon, (-\mathcal{S}) f_\varepsilon \gg_\star$$

and using (75),

$$- \ll \mathcal{L}^{\mathbf{m}} h_z, f_\varepsilon \gg_\star + z \ll h_z, f_\varepsilon \gg_\star = - \ll f_\varepsilon, g_\varepsilon \gg_\star - \ll f_\varepsilon, (-\mathcal{S}) f_\varepsilon \gg_\star .$$

First, let z go to 0, and observe that the limit of $\ll \mathcal{L}^{\mathbf{m}} h_z, f_\varepsilon \gg_\star$ exists since $\{\mathcal{L}^{\mathbf{m}} h_z\}$ weakly converges in $\mathbb{H}_{-\mathbb{1}\star}$ and $f_\varepsilon \in \mathbb{H}_{\mathbb{1}\star}$. Let us take the limit as ε goes to 0, and write

$$\ll f_\varepsilon, g_\varepsilon \gg_\star \leqslant \| f_\varepsilon \|_{\mathbb{1}\star} \, \| g_\varepsilon \|_{-\mathbb{1}\star} \leqslant \mathrm{C} |||g_\varepsilon|||_1 \xrightarrow[\varepsilon \to 0]{} 0.$$

The first equality is justified by the fact that g_ε belongs to $\mathcal{Q}_0 \subset \mathbb{H}_{-\mathbb{1}\star}$, and the last inequality comes from the definition of the semi-norm $|||\cdot|||_1$ given in (27). As a consequence, we have obtained

$$\lim_{\varepsilon \to 0} \ll f_\varepsilon, (-\mathcal{S}) f_\varepsilon \gg_\star = \lim_{\varepsilon \to 0} \lim_{z \to 0} \ll \mathcal{L}^{\mathbf{m}} h_z, f_\varepsilon \gg_\star .$$

In the same way, take the inner product $\ll \cdot, \cdot \gg_\star$ of (76) with h_z to obtain

$$\ll j^{\mathrm{A}}_{0,1}, h_z \gg_\star = - \ll g_\varepsilon, h_z \gg_\star + \ll \mathcal{L}^{\mathbf{m},\star} f_\varepsilon, h_z \gg_\star .$$

If we send first z to 0, then $\ll g_\varepsilon, h_z \gg_\star$ converges to $\ll g_\varepsilon, h_0 \gg_\star$ from the weak convergence of $\{h_z\}$ in $\mathbb{H}_{\mathbb{1}\star}$ and since $g_\varepsilon \in \mathbb{H}_{-\mathbb{1}\star}$. As before, we write

$$\ll g_\varepsilon, h_0 \gg_\star \leqslant \mathrm{C} |||g_\varepsilon|||_1 \xrightarrow[\varepsilon \to 0]{} 0.$$

Therefore,

$$\lim_{z \to 0} \ll j^{\mathrm{A}}_{0,1}, h_z \gg_\star = \lim_{\varepsilon \to 0} \lim_{z \to 0} \ll \mathcal{L}^{\mathbf{m},\star} f_\varepsilon, h_z \gg_\star = \lim_{\varepsilon \to 0} \ll f_\varepsilon, (-\mathcal{S}) f_\varepsilon \gg_\star$$

and the claim is proved. $\square$

8 The anharmonic chain perturbed by a diffusive noise

In this section we say a few words about the anharmonic chain, when the interactions between atoms are non-linear, given by a potential V. As in [23], we assume that the function $\mathrm{V}:\mathbb{R}\to\mathbb{R}_+$ satisfies the following properties:

(i) $\mathrm{V}(\cdot)$ is a smooth symmetric function,

(ii) there exist δ_- and δ_+ such that $0<\delta_-\leqslant \mathrm{V}''(\cdot)\leqslant\delta_+<+\infty$,

(iii) $\delta_-/\delta_+>(3/4)^{1/16}$.

Using the same notations as in the introduction, the configuration $\{p_x,r_x\}$ now evolves according to

$$\begin{cases} \dfrac{\mathrm{d}p_x}{\mathrm{d}t} = \mathrm{V}'(r_{x+1})-\mathrm{V}'(r_x), \\ \dfrac{\mathrm{d}r_x}{\mathrm{d}t} = \dfrac{p_x}{\mathrm{M}_x}-\dfrac{p_{x-1}}{\mathrm{M}_{x-1}}. \end{cases} \tag{77}$$

We define $\pi_x := p_x/\sqrt{\mathrm{M}_x}$, and the dynamics on $\{\pi_x,r_x\}$ rewrites as:

$$\begin{cases} \dfrac{\mathrm{d}\pi_x}{\mathrm{d}t} = \dfrac{1}{\sqrt{\mathrm{M}_x}}\big[\mathrm{V}'(r_{x+1})-\mathrm{V}'(r_x)\big], \\ \dfrac{\mathrm{d}r_x}{\mathrm{d}t} = \dfrac{\pi_x}{\sqrt{\mathrm{M}_x}}-\dfrac{\pi_{x-1}}{\sqrt{\mathrm{M}_{x-1}}}. \end{cases} \tag{78}$$

The total energy

$$\mathcal{E} := \sum_{x\in\mathbb{Z}}\left\{\frac{\pi_x^2}{2}+\mathrm{V}(r_x)\right\}$$

is conserved. The flip and exchange noises have poor ergodic properties, and can be used for harmonic chains only. For the anharmonic case, we introduce a stronger stochastic perturbation. Now, the total generator of the dynamics writes $\mathcal{L}^{\mathbf{m}}=\mathcal{A}^{\mathbf{m}}+\gamma\mathcal{S}$, where

$$\mathcal{A}^{\mathbf{m}} := \sum_x \frac{1}{\sqrt{\mathrm{M}_x}}\big(\mathrm{X}_x-\mathrm{Y}_{x,x+1}\big), \qquad \mathcal{S} := \frac{1}{2}\sum_x\big\{\mathrm{X}_x^2+\mathrm{Y}_{x,x+1}^2\big\}, \tag{79}$$

where

$$\mathrm{Y}_{x,y} = \pi_x\frac{\partial}{\partial r_y}-\mathrm{V}'(r_y)\frac{\partial}{\partial \pi_x}, \qquad \mathrm{X}_x=\mathrm{Y}_{x,x}.$$

For this anharmonic case, the needed ingredients can be proved directly from [23]: first, note that the symmetric part of the generator does not depend on the disorder and is exactly the same as in [23]. Then, the proof of the spectral gap is done in Section 12 of that paper, and the sector condition can also be proved, following Section 8. More precisely, after taking into account the disorder and its fluctuation, the same argument of [23, Lemma 8.2, Section 8] can be applied, since it is mainly based on the fact that both antisymmetric and symmetric parts involve the same operators $\mathrm{Y}_{x,y}$.

9 Hydrodynamic limits

We briefly discuss the failure in the derivation of the hydrodynamic limits. Let us assume that the initial law for the Markov process $\{\omega(t)\}_{t\geqslant 0}$ (still generated by $\mathrm{N}^2\mathcal{L}_\mathrm{N}^{\mathbf{m}}$), is not the equilibrium measure μ_β^N, but a *local equilibrium measure* (see (81) below), and fix the disorder $\mathbf{m}$. The main goal would be to prove that this property of local equilibrium propagates in time: in other words hydrodynamics limits hold, with an energy profile solution to the diffusion equation with constant coefficient D.

9.1 Statement of the hydrodynamic limits conjecture

Let us fix once more some notations. The distribution at time t of the Markov chain on $\mathbb{T}_N$ with the generator $N^2\mathcal{L}_N^{\mathbf{m}}$ and the initial probability measure μ^N is denoted by $\mathbb{P}_{N,t}^{\mathbf{m}}$. The measure induced by $\mathbb{P}_{N,t}^{\mathbf{m}}$ on $\mathcal{D}([0,T],\Omega_N)$ is denoted by $\mathbb{P}_N^{\mathbf{m}}$. The set of probability measures on $\mathbb{T}$, denoted by $\mathcal{M}_1$, is endowed with the weak topology. We also introduce $\mathcal{D}([0,T],\mathcal{M}_1)$ namely the Skorokhod space of trajectories in $\mathcal{M}_1$. The measure induced by $\mathbb{P}_N^{\mathbf{m}}$ on $\mathcal{D}([0,T],\mathcal{M}_1)$ is denoted by $\mathcal{Q}_N^{\mathbf{m}} := \mathbb{P}_N^{\mathbf{m}} \circ (\pi^N)^{-1}$, where

$$\pi^N := \frac{1}{N}\sum_{x\in\mathbb{T}_N} \omega_x^2 \delta_{x/N}.$$

Expectation with respect to $\mathbb{P}_N^{\mathbf{m}}$ is denoted by $\mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}}$.

CONJECTURE 9.1. *Let $T > 0$ be a time-horizon. Let $\{\mu^N\}_N$ be a sequence of probability measures on Ω_N. Under suitable conditions on the initial law μ^N, for almost every realization of the disorder $\mathbf{m}$, the measure $\mathcal{Q}_N^{\mathbf{m}}$ weakly converges in $\mathcal{D}([0,T],\mathcal{M}_1)$ to the probability measure concentrated on the path $\{\mathbf{e}(t,u)du\}_{t\in[0,T]}$, where $\mathbf{e}$ is the unique weak solution to the system*

$$\begin{cases} \dfrac{\partial \mathbf{e}}{\partial t}(t,u) = D\dfrac{\partial^2 \mathbf{e}}{\partial u^2}(t,u), & t>0,\ u\in\mathbb{T} \\ \mathbf{e}(0,u) = \mathbf{e}_0(u). \end{cases}$$

What we expect as for “suitable assumptions” on the initial law are the common ones in the literature of hydrodynamic limits, when dealing with non compact spaces. The first one is natural and related on the relative entropy:

ASSUMPTION 9.2. *Let us assume that there exists a positive constant K_0 such that the relative entropy $H(\mu^N|\mu_\star^N)$ of μ^N with respect to some reference measure $\mu_\star^N$ (for example the Gibbs state with temperature $\beta^{-1} = 1$) is bounded by K_0N:*

$$H(\mu^N|\mu_\star^N) \leqslant K_0 N. \tag{80}$$

For instance, if μ^N is defined as a Gibbs local equilibrium state:

$$\prod_{x\in\mathbb{T}_N} \sqrt{\frac{\beta_0(x/N)}{2\pi}} \exp\left(-\frac{\beta_0(x/N)}{2}\omega_x^2\right) d\omega_x \tag{81}$$

for some continuous function $\beta_0 : \mathbb{T} \to \mathbb{R}_+$, then (80) is satisfied. The second one is related to energy bounds, which have already been a major concern in [29]. More precisely,

ASSUMPTION 9.3. *We assume that there exists a positive constant E_0 such that*

$$\limsup_{N\to\infty} \mu^N\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N}\omega_x^4\right] \leqslant E_0. \tag{82}$$

In the derivation of hydrodynamic limits with the usual *entropy method*, we need the following two estimates: first, there exists a positive constant C such that, for any $t > 0$

$$\mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}}\left[\frac{1}{N}\sum_{x\in\mathbb{T}_N}\omega_x^2(t)\right] \leqslant C. \tag{83}$$

This can be easily established using (82) and the Cauchy-Schwarz inequality. The second control that we need is

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}}\left[\int_0^t \frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\omega_x^4(s)\,ds\right]=0. \tag{84}$$

If μ^N is a convex combination of Gibbs local equilibrium states, then the same argument of [29] shows that the law of the process remains a convex combination of Gaussian measures, and that (84) holds.

Contrary to the velocity-flip model, we do not need to assume a good control of every energy moment if we expect the usual entropy method to work. This technical need was only due to the relative entropy method.

With Assumptions 9.2 and 9.3 we could try to prove Conjecture 9.1 by using the entropy method, which permits to consider general initial profiles (for example, the profile β_0 can be assumed only bounded, not smooth). The usual technical points of this well-known procedure are the one and two-blocks estimates, as well as tightness. In this model, they are somehow easy to prove because the diffusion coefficient is constant, and there is no need to show its regularity.

9.2 Replacement of the current by a gradient

In this subsection we recall the main steps of the usual entropy method, and explain which ones can be proved for our system. We fix the disorder $\mathbf{m}=\{m_x\}_{x\in\mathbb{T}_N}$ and $T>0$. For $t\in[0,T]$, we denote by $\mathcal{Z}_{t,\mathbf{m}}^N$ the empirical energy field defined as

$$\mathcal{Z}_{t,\mathbf{m}}^N(H)=\frac{1}{N}\sum_{x\in\mathbb{T}_N} H\left(\frac{x}{N}\right)\omega_x^2(t),$$

where $H:\mathbb{T}\to\mathbb{R}$ is a smooth function. We rewrite $\mathcal{Z}_{t,\mathbf{m}}^N(H)$ as in Section 3.3 as

$$\mathcal{Z}_{t,\mathbf{m}}^N(H)=\mathcal{Z}_{0,\mathbf{m}}^N(H)+\int_0^t\sum_{x\in\mathbb{T}_N}\nabla_N H\left(\frac{x}{N}\right) j_{x,x+1}(\mathbf{m},s)ds+\mathfrak{M}_{t,\mathbf{m}}^N(H),$$

where $\mathfrak{M}_{t,\mathbf{m}}^N(H)$ is a martingale. The strategy consists in replacing the current $j_{x,x+1}$ by the linear combination given in Theorem 5.9. For that purpose, for any $f\in\mathcal{Q}$ we rewrite

$$\mathcal{Z}_{t,\mathbf{m}}^N(H)=\mathcal{Z}_{0,\mathbf{m}}^N(H)+\int_0^t D\mathcal{Z}_{s,\mathbf{m}}^N(\Delta_N H)ds+\mathfrak{I}_{t,\mathbf{m},f}^{1,N}(H)+\mathfrak{I}_{t,\mathbf{m},f}^{2,N}(H)+\mathfrak{M}_{t,\mathbf{m}}^N(H),$$

where

$$\mathfrak{I}_{t,\mathbf{m},f}^{1,N}(H)=\int_0^t\sum_{x\in\mathbb{T}_N}\nabla_N H\left(\frac{x}{N}\right)\left[j_{x,x+1}(\mathbf{m},s)+D(\omega_{x+1}^2-\omega_x^2)(s)+\mathcal{L}_N^{\mathbf{m}}(\tau_x f)(\mathbf{m},s)\right]ds,$$

$$\mathfrak{I}_{t,\mathbf{m},f}^{2,N}(H)=-\int_0^t\sum_{x\in\mathbb{T}_N}\nabla_N H\left(\frac{x}{N}\right)\mathcal{L}_N^{\mathbf{m}}(\tau_x f)(\mathbf{m},s)ds.$$

Theorem 9.1 would follow from the three lemmas below.

Lemma 9.4. *For every $\mathbf{m}\in\Omega_{\mathcal{D}}$, for every smooth function $H:\mathbb{T}\to\mathbb{R}$ and every $\delta>0$,*

$$\lim_{N\to\infty}\mathbb{P}_N^{\mathbf{m}}\left[\sup_{[0,T]}\left|\mathfrak{M}_{t,\mathbf{m}}^N(H)\right|>\delta\right]=0.$$

Lemma 9.5. *For every $\mathbf{m} \in \Omega_{\mathcal{D}}$, for every $f \in \mathcal{Q}$ and every smooth function $\mathrm{H} : \mathbb{T} \to \mathbb{R}$,*

$$\limsup_{\mathrm{N}\to\infty} \mathbb{E}_{\mathbb{P}_{\mathrm{N}}^{\mathbf{m}}} \left[\left| \mathfrak{I}^{2,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) \right| \right] = 0.$$

Lemma 9.6. *There exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \in \mathcal{Q}$ such that, for every smooth function $\mathrm{H} : \mathbb{T} \to \mathbb{R}$,*

$$\lim_{k\to\infty} \lim_{\mathrm{N}\to\infty} \mathbb{E}\left[\mathbb{E}_{\mathbb{P}_{\mathrm{N}}^{\mathbf{m}}} \left[\left| \mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f_k}(\mathrm{H}) \right| \right] \right] = 0.$$

Lemma 9.4 and Lemma 9.5 can be proved, following the same standard arguments given for example in [16, Section 7]. We need the energy moment estimate (84) in Lemma 9.4, in the computation of the quadratic variation of the martingale. The next subsection is devoted to highlight what fails in Lemma 9.6, which should be related to the results of Sections 4, 5 and 6.

Remark 9.1. Conditioned to proving Lemma 9.6, Theorem 9.1 would follow: recall that $\mathcal{Q}_{\mathrm{N}}^{\mathbf{m}}$ is the distribution on the path space $\mathcal{D}([0,\mathrm{T}], \mathcal{M}_1)$ of the process π_t^{N}. Following the same argument as for the generalized exclusion process in [16, Section 7.6], we can show that the sequence $\left\{\mathcal{Q}_{\mathrm{N}}^{\mathbf{m}}, \mathrm{N} \geqslant 1\right\}$ is weakly relatively compact. It remains to prove that every limit point $\mathcal{Q}_*^{\mathbf{m}}$ is concentrated on absolutely continuous paths $\mathbf{e}(t, du) = \mathbf{e}(t,u)du$ whose densities are solutions to the hydrodynamic equations given in Theorem 9.1. It could be seen from Lemma 9.6 by following the proof of [16, Theorem 7.0.1].

9.3 Failed variance estimate

In this paragraph we fix the disorder $\mathbf{m}$, and we erase it whenever no confusion arises. We are going to recall here the usual main steps of the entropy method. We rewrite $\mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H})$ as

$$\mathfrak{I}^{1,\mathrm{N}}_{t,\mathbf{m},f}(\mathrm{H}) = \int_0^t \sum_{x\in\mathbb{T}_{\mathrm{N}}} \mathrm{G}\left(\frac{x}{\mathrm{N}}\right) \tau_x \varphi(\mathbf{m}, s) ds,$$

where

$$\begin{cases} \varphi(\mathbf{m}, s) := \varphi(\mathbf{m}, \omega(s)) := j_{0,1}(\mathbf{m}, \omega(s)) + \mathrm{D}(\omega_1^2 - \omega_0^2)(s) + \mathcal{L}_{\mathrm{N}}^{\mathbf{m}}(f)(\mathbf{m}, \omega(s)) \\ \mathrm{G}\left(\frac{x}{\mathrm{N}}\right) := \nabla_{\mathrm{N}} \mathrm{H}\left(\frac{x}{\mathrm{N}}\right). \end{cases}$$

Entropy inequality – In Lemma 9.6, note that the expectation with respect to the law of the process $\mathbb{P}_{\mathrm{N}}^{\mathbf{m}}$ is taken. There is *a priori* no hope to get any estimate of this expectation, apart from the well-known entropy inequality. More precisely, let us denote by $\mathrm{X}_{\mathrm{N}}^f(\omega)$ the following quantity:

$$\mathrm{X}_{\mathrm{N}}^f(\omega) := \sum_{x\in\mathbb{T}_{\mathrm{N}}} \mathrm{G}\left(\frac{x}{\mathrm{N}}\right) \tau_x \varphi(\omega).$$

From the entropy inequality, we obtain

$$\mathbb{E}_{\mathbb{P}_{\mathrm{N}}^{\mathbf{m}}} \left[\left| \int_0^{\mathrm{T}} \mathrm{X}_{\mathrm{N}}^f(\omega)(s)\, ds \right| \right] \leqslant \frac{1}{\alpha \mathrm{N}} \mathrm{H}(\mathbb{P}_{\mathrm{N}}^{\mathbf{m}} | \mu_\beta^{\mathrm{N}}) + \frac{1}{\alpha \mathrm{N}} \log \mathbb{E}_{\mu_\beta^{\mathrm{N}}} \left[\exp\left(\alpha \mathrm{N} \left| \int_0^{\mathrm{T}} \mathrm{X}_{\mathrm{N}}^f(\omega)(s)\, ds \right| \right) \right],$$

for all $\alpha > 0$. Since the entropy is decreasing in time, we know that, for all disorder field $\mathbf{m}$, $\mathrm{H}(\mathbb{P}_{\mathrm{N}}^{\mathbf{m}} | \mu_\beta^{\mathrm{N}})$ is bounded. From the arbitrariness of α, we are reduced to investigate the convergence of the second term in the previous right hand side.

Feynman-Kac formula – Usually, the purpose is to reduce the dynamics problem to the study of the largest eigenvalue for a small perturbation of the generator $N^2\mathcal{S}_N$. This reduction relies on the Feynman-Kac formula together with variational formula for the largest eigenvalue of a symmetric operator. By Feynman-Kac formula,

$$\mathbb{E}_{\mu_\beta^N}\left[\exp\left\{N\int_0^T X_N^f(\omega)(s)\,ds\right\}\right] \leqslant \exp\left\{\int_0^T \lambda_N(s)\,ds\right\}$$

where $\lambda_N(s)$ is the largest eigenvalue of the symmetric operator $N^2\mathcal{S}_N(\cdot)+NX_N^f(\omega)$. From the variational formula for the largest eigenvalue of an operator in a Hilbert space, we also know that

$$\lambda_N(s) \leqslant \sup_g\left\{\left\langle NX_N^f(\cdot)\,g(\cdot)\right\rangle_\beta - N^2\mathcal{D}_N(\mu_\beta;\sqrt{g})\right\}$$

where the supremum is taken over all measurable functions g which are densities with respect to μ_β^N. In particular,

$$\frac{1}{N}\log\mathbb{E}_{\mu_\beta^N}\left[\exp\left\{\int_0^T NX_N^f(\omega)(s)\,ds\right\}\right] \leqslant \int_0^T \sup_g\left\{\left\langle X_N^f(\omega)\;g(\omega)\right\rangle_\beta - N\mathcal{D}_N(\mu_\beta;\sqrt{g})\right\}ds.$$

Reduction to microscopic blocks – With the same spirit of the one-block estimate presented in [29], it is then crucial to replace microscopic quantities with their spatial averages. Here, with the same ideas of [16], we can replace

$$\begin{array}{lll} j_{0,1} & \text{by} & \dfrac{1}{2(\ell-1)+1}\displaystyle\sum_{x\in\Lambda_{\ell-1}} j_{x,x+1} \\ \omega_0^2 & \text{by} & \dfrac{1}{2\ell+1}\displaystyle\sum_{x\in\Lambda_\ell}\omega_x^2 \\ \mathcal{L}_N^{\mathbf{m}}(f)(\omega) & \text{by} & \dfrac{1}{2\ell_f+1}\displaystyle\sum_{x\in\Lambda_{\ell_f}}\mathcal{L}_{s_f+1}^{\mathbf{m}}(\tau_x f) \end{array}$$

where $\ell_f = \ell - s_f - 1$ so that $\mathcal{L}_{s_f+1}(\tau_y f)$ is $\mathcal{F}_{\Lambda_\ell}$-mesurable for every $y\in\Lambda_{\ell_f}$. Let us introduce the following notation

$$W^{f,\ell} := \frac{1}{2\ell'+1}\sum_{x\in\Lambda_{\ell'}} j_{x,x+1} + \frac{D}{2\ell+1}\sum_{x\in\Lambda_\ell}(\omega_{x+1}^2-\omega_x^2) + \frac{1}{2\ell_f+1}\sum_{x\in\Lambda_{\ell_f}}\mathcal{L}_{s_f+1}(\tau_x f) \tag{85}$$

with $\ell' = \ell-1$. Finally, thanks to the regularity of the function G and the fact that D is constant, we are able to reduce Lemma 9.6 to Lemma 9.7 below. We also need to perform a cut-off in order to control high energy values, and this is valid thanks to (84).

Lemma 9.7. *For all $\delta>0$,*

$$\inf_{f\in\mathcal{Q}}\limsup_{\ell\to\infty}\limsup_{N\to\infty}\sup_g\left\{\left\langle Y_{N,\ell}^f(\omega)\,g(\omega)\right\rangle_\beta - \delta N\mathcal{D}_N(\mu_\beta;\sqrt{g})\right\}\leqslant 0, \tag{86}$$

where

$$Y_{N,\ell}^f(\omega) := \sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right)\tau_x W^{f,\ell}(\omega).$$

Reduction to a variance estimate – Then, the challenge is to reduce the proof of Lemma 9.7 to the following result:

$$\inf_{f\in\mathcal{Q}}\lim_{\ell\to\infty}(2\ell)\times\mathbb{E}\left[\left\langle\left(-\mathcal{S}_{\Lambda_\ell}\right)^{-1}\mathrm{W}^{f,\ell},\mathrm{W}^{f,\ell}\right\rangle_\beta\right]=0 \tag{87}$$

This convergence holds, since it is equivalent to the conclusion of Theorem 5.9, where the diffusion coefficient D is defined through the non-gradient approach. Here is the main obstacle. If we follow the strategy given in [16, Section 7.3], we can bound the supremum in (86) by the largest eigenvalue of $\mathcal{S}_{\Lambda_\ell}+\mathbf{b}\mathrm{W}^{f,\ell}$ where $\mathbf{b}$ is a small constant. In order to estimate this largest eigenvalue, we usually use a perturbation method which provides a bound on the largest eigenvalue in terms of the variance of $\mathrm{W}^{f,\ell}$. This can not be proved, and suggests that the entropy inequality together with the Feynman-Kac formula are not the good tools to prove the hydrodynamic limits for systems which do not have a spectral gap (see the last concluded section).

We conclude this section by explaining why the perturbation theory does not work. Let us try to prove Lemma 9.7. Since μ_β is translation invariant, we may rewrite $\left\langle \mathrm{Y}_{\mathrm{N}}^{f,\ell}(\cdot)\,g(\cdot)\right\rangle_\beta$ as

$$\sum_{x\in\mathbb{T}_\mathrm{N}}\left\langle \mathrm{G}\left(\frac{x}{\mathrm{N}}\right)\mathrm{W}^{f,\ell}(\omega)\tau_{-x}g(\omega)\right\rangle_\beta .$$

Since the Dirichlet form is convex, the supremum in (86) is bounded from above by

$$\frac{\delta\mathrm{N}}{2\ell}\sum_{x\in\mathbb{T}_\mathrm{N}}\sup_g\left\{\mathbf{b}\left\langle\mathrm{W}^{f,\ell}\,g\right\rangle_\beta-\mathcal{D}_\ell(\mu_\beta;\sqrt{g})\right\}, \tag{88}$$

where the constant $\mathbf{b}=\mathbf{b}(x,\ell,\delta,\mathrm{N})$ satisfies

$$|\mathbf{b}|:=\left|\mathrm{G}\left(\frac{x}{\mathrm{N}}\right)\frac{2\ell}{\delta\mathrm{N}}\right|\leqslant\|\mathrm{G}\|_\infty\frac{2\ell}{\delta\mathrm{N}}.$$

Let us denote by $\lambda_{\mathrm{N},\ell,f}$ this last supremum inside the sum (88), which does not depend on x. We consider a sequence $\{g_k\}_{k\in\mathbb{N}}$ that approaches this supremum, such that

$$\lim_{k\to\infty}\left\langle\sqrt{g_k},\left(\mathcal{S}_{\Lambda_\ell}+\mathbf{b}\mathrm{W}^{f,\ell}\right)\sqrt{g_k}\right\rangle_\beta=\lambda_{\mathrm{N},\ell,f}.$$

The idea of the perturbation theory is to expand $\sqrt{g_k}$ around the constant value 1. We write

$$\left\langle\sqrt{g_k},(\mathcal{S}_{\Lambda_\ell}+\mathbf{b}\mathrm{W}^{f,\ell})\sqrt{g_k}\right\rangle_\beta=\mathbf{b}\Big(\left\langle\mathrm{W}^{f,\ell}\right\rangle_\beta+2\left\langle\mathrm{W}^{f,\ell}(\sqrt{g_k}-1)\right\rangle_\beta+\left\langle\mathrm{W}^{f,l}(\sqrt{g_k}-1)^2\right\rangle_\beta\Big)\\ -\mathcal{D}_\ell(\mu_\beta;\sqrt{g_k}). \tag{89}$$

We know that $\left\langle\mathrm{W}^{f,\ell}\right\rangle_\beta=0$, and we use the Cauchy-Schwarz inequality for the scalar product $\langle\cdot,(-\mathcal{S}_{\Lambda_\ell})\cdot\rangle_\beta$ in the second term. We obtain that (89) is bounded, for every $\mathrm{A}>0$, by

$$\mathbf{b}\left(\frac{\mathbf{b}}{\mathrm{A}}\left\langle\mathrm{W}^{f,\ell},(-\mathcal{S}_{\Lambda_\ell})^{-1}\mathrm{W}^{f,\ell}\right\rangle_\beta+\frac{\mathrm{A}}{\mathbf{b}}\mathcal{D}_\ell(\mu_\beta;\sqrt{g_k})\right)+\mathbf{b}\left\langle\mathrm{W}^{f,\ell}(\sqrt{g_k}-1)^2\right\rangle_\beta-\mathcal{D}_\ell(\mu_\beta;\sqrt{g_k}).$$

It remains to bound the third term in the expression above. This could be done if we had the following lemma.

Lemma 9.8. *There exists a constant* $\mathrm{C}:=\mathrm{C}(\ell,f,\beta,\gamma,\lambda)$ *such that, for every* $g\geqslant 0$,

$$\left\langle\mathrm{W}^{f,\ell}(\sqrt{g}-1)^2\right\rangle_\beta\leqslant\mathrm{C}\mathcal{D}_\ell(\mu_\beta;\sqrt{g}). \tag{90}$$

As before, we could try to use the fact that $W^{f,\ell}$ is a quadratic function. Even this fact is not helpful, and we give now a counter-example to this last lemma. We denote by H_n the normalized one-variable Hermite polynomial of degree $n \geqslant 3$ (see Appendix A). Let us consider

$$\begin{cases} \sqrt{g}(\omega) = |H_n(\omega_0)| \\ W^{f,\ell}(\omega) = H_2(\omega_0) = \omega_0^2 - 1. \end{cases}$$

Let us note that $\langle H_n^2 \rangle_\beta = 1$, and $\langle H_2 \rangle_\beta = 0$, so that the two test functions g and $W^{f,\ell}$ satisfy all expected conditions. By using the recursive relation

$$H_{n+1}(\omega_0) = \omega_0 H_n(\omega_0) - n H_{n-1}(\omega_0),$$

we get for the left hand side of (90),

$$\begin{aligned} \big\langle H_2(|H_n| - 1)^2 \big\rangle_\beta &= \big\langle \omega_0^2 H_n^2(\omega_0) \big\rangle_\beta - \big\langle H_n^2 \big\rangle_\beta - 2\big\langle H_2 |H_n| \big\rangle_\beta + \big\langle H_2 \big\rangle_\beta \\ &= \big\langle H_{n+1}^2 + 2n H_{n+1} H_{n+1} + n^2 H_{n-1}^2 \big\rangle_\beta - 1 - 2\big\langle H_2 |H_n| \big\rangle_\beta \\ &= 1 + n^2 - 1 - 2\big\langle H_2 |H_n| \big\rangle_\beta \geqslant n^2 - 2. \end{aligned}$$

Above the last equality comes from the orthonormality of the polynomial basis, and the last inequality is a consequence of the Cauchy-Schwarz inequality $\langle H_2 |H_n| \rangle_\beta^2 \leqslant \langle H_2^2 \rangle_\beta \langle H_n^2 \rangle_\beta = 1$. Let us assume that there exists a constant $C > 0$ which does not depend on n such that

$$n^2 - 2 \leqslant \big\langle H_2(|H_n| - 1)^2 \big\rangle_\beta \leqslant C \mathcal{D}_\ell(\mu_\beta; |H_n|).$$

From the convexity of the Dirichlet form, we have

$$\mathcal{D}_\ell(\mu_\beta; |H_n|) \leqslant \mathcal{D}_\ell(\mu_\beta; H_n).$$

In the case where n is an even positive integer, the flip noise gives a zero contribution to the Dirichlet form, and then, for all n even, we have

$$\mathcal{D}_\ell(\mu_\beta; H_n) = \frac{\lambda}{2} \Big\langle \big(H_n(\omega_1) - H_n(\omega_0)\big)^2 \Big\rangle_\beta = \lambda \langle H_n^2 \rangle_\beta - \lambda \big\langle H_n(\omega_0) H_n(\omega_1) \big\rangle_\beta = \lambda.$$

In the last equality, we use the fact that H_n is unitary, and that $H_n(\omega_0)H_n(\omega_1)$ constitutes another element of the Hermite polynomial basis, then is orthogonal to the constant polynomial 1. Letting n go to infinity, we obtain a contradiction to (90).

Ergodic decomposition – Another idea would be to use the ergodic decomposition. The generator $\mathcal{S}_{\Lambda_\ell}$ restricted to finite boxes does not have a spectral gap, but it becomes ergodic when restricted to some finite orbits. However, this approach fails, because the space is not compact, and we need to disintegrate the measure μ_β with respect to all energy levels in $(0, +\infty)$. This enforces us to introduce a cut-off in the variational formula giving the largest eigenvalue. In other words, an indicator function $\mathbf{1}\{|\omega_x| \leqslant E_0\}$ will appear in front of $W^{f,\ell}$. Finally, we will have to deal with functions of the configurations that are not quadratic any more, and we do not know how to prove the convergence result (87) for general functions.

9.4 Conclusion

Even if the non-gradient method can be applied in some cases when the spectral gap does not hold, (and then the diffusion coefficient is well defined), this does not straightforwardly imply the hydrodynamic limits.

In order to derive the hydrodynamic theorem, we would need to bypass the entropy inequality together with the Feynman-Kac formula. The entropy inequality is however a convenient mean to transform the averages w.r.t. the unknown law μ_t^N into equilibrium averages w.r.t. μ_β^N, which are more easily tractable. The same problem would arise in the relative entropy method.

A Hermite polynomials and quadratic functions

In the whole section we assume $\beta = 1$. Every result can be restated for the general case after replacing the variable ω by $\beta^{-1/2}\omega$.

Let χ be the set of positive integer-valued functions $\xi : \mathbb{Z} \to \mathbb{N}$, such that ξ_x vanish for all but a finite number of $x \in \mathbb{Z}$. The *length* of ξ, denoted by $|\xi|$, is defined as

$$|\xi| = \sum_{x \in \mathbb{Z}} \xi_x.$$

For $\xi \in \chi$, we define the polynomial function H_ξ on Ω as

$$\mathrm{H}_\xi(\omega) = \prod_{x \in \mathbb{Z}} h_{\xi_x}(\omega_x),$$

where $\{h_n\}_{n \in \mathbb{N}}$ are the normalized Hermite polynomials w.r.t. the one-dimensional standard Gaussian probability law (with density $(2\pi)^{-1/2}\exp(-x^2/2)$ on $\mathbb{R}$). The sequence $\{\mathrm{H}_\xi\}_{\xi \in \chi}$ forms an orthonormal basis of the Hilbert space $\mathbf{L}^2(\mu_1)$, where μ_1 is the infinite product Gibbs measure on $\mathbb{R}^{\mathbb{Z}}$, defined by (4) with $\beta = 1$. As a result, every function $f \in \mathbf{L}^2(\mu_1)$ can be decomposed in the form

$$f(\omega) = \sum_{\xi \in \chi} \mathrm{F}(\xi)\mathrm{H}_\xi(\omega).$$

Moreover, we can compute the scalar product $\langle f, g\rangle_1$ for $f = \sum_\xi \mathrm{F}(\xi)\mathrm{H}_\xi$ and $g = \sum_\xi \mathrm{G}(\xi)\mathrm{H}_\xi$ as

$$\langle f, g\rangle_1 = \sum_{\xi \in \chi} \mathrm{F}(\xi)\mathrm{G}(\xi).$$

DEFINITION A.1. *We denote by $\chi_n \subset \chi$ the subset sequences of length n, i.e. $\chi_n := \{\xi \in \chi \; ; \; |\xi| = n\}$. A function $f \in \mathbf{L}^2(\mu_1)$ is of degree n if its decomposition*

$$f = \sum_{\xi \in \chi} \mathrm{F}(\xi)\mathrm{H}_\xi$$

satisfies: $\mathrm{F}(\xi) = 0$ *for all* $\xi \notin \chi_n$.

REMARK A.1. In this paper, we mainly focus on degree 2 functions, which are by Definition A.1 of the form

$$\sum_{x \in \mathbb{Z}} \varphi(x,x)(\omega_x^2 - 1) + \sum_{x \neq y} \varphi(x,y)\omega_x\omega_y \tag{91}$$

where $\varphi : \mathbb{Z}^2 \to \mathbb{R}$ is a square summable symmetric function. Note that they all have zero mean w.r.t. μ_1, and they can also be rewritten as

$$\sum_{x \in \mathbb{Z}} \psi(x,x)(\omega_x^2 - \omega_{x+1}^2) + \sum_{x \neq y} \psi(x,y)\omega_x\omega_y,$$

for some square summable symmetric function $\psi : \mathbb{Z}^2 \to \mathbb{R}$.

A.1 Local functions

On the set of n-tuples $\mathbf{x} := (x_1, \dots, x_n)$ of $\mathbb{Z}^n$, we introduce the equivalence relation $\mathbf{x} \sim \mathbf{y}$ if there exists a permutation σ on $\{1, \dots, n\}$ such that $x_{\sigma(i)} = y_i$ for all $i \in \{1, \dots, n\}$. The class of $\mathbf{x}$ for the relation $\sim$ is denoted by $[\mathbf{x}]$ and its cardinal by $c(\mathbf{x})$. Then the set of configurations of χ_n can be identified with the set of n-tuples classes for $\sim$ by the one-to-one application:

$$\begin{aligned} \mathbb{Z}^n/\sim \quad &\to \chi_n \\ [\mathbf{x}] = [(x_1, \dots, x_n)] \mapsto \quad &\xi^{[\mathbf{x}]} \end{aligned}$$

where for any $y \in \mathbb{Z}$, $(\xi^{[\mathbf{x}]})_y = \sum_{i=1}^n \mathbf{1}_{y=x_i}$. We identify $\xi \in \chi_n$ with the occupation numbers of a configuration with n particles, and $[\mathbf{x}]$ corresponds to the positions of those n particles. A function $\mathrm{F} : \chi_n \to \mathbb{R}$ is nothing but a symmetric function $\mathrm{F} : \mathbb{Z}^n \to \mathbb{R}$ through the identification of ξ with $[\mathbf{x}]$. We denote (with some abuse of notations) by $\langle \cdot, \cdot \rangle$ the scalar product on $\oplus \mathbf{L}^2(\chi_n)$, each χ_n being equipped with the counting measure. Hence, for two functions $\mathrm{F}, \mathrm{G} : \chi \to \mathbb{R}$, we have

$$\langle \mathrm{F}, \mathrm{G} \rangle = \sum_{n \geqslant 0} \sum_{\xi \in \chi_n} \mathrm{F}_n(\xi) \mathrm{G}_n(\xi) = \sum_{n \geqslant 0} \sum_{\mathbf{x} \in \mathbb{Z}^n} \frac{1}{c(\mathbf{x})} \mathrm{F}_n(\mathbf{x}) \mathrm{G}_n(\mathbf{x}),$$

with $\mathrm{F}_n, \mathrm{G}_n$ the restrictions of F, G to χ_n.

A.2 Dirichlet form

It is not hard to check the following proposition, which is a direct consequence of the fact that h_n has the same parity of the integer n.

Proposition A.1. *If a local function $f \in \mathbf{L}^2(\mu_1)$ is written on the form $f = \sum_{\xi \in \chi} \mathrm{F}(\xi) \mathrm{H}_\xi$, then*

$$\mathcal{S} f(\omega) = \sum_{\xi \in \chi} (\mathfrak{S}\mathrm{F})(\xi) \mathrm{H}_\xi(\omega),$$

where $\mathfrak{S}$ is the operator acting on functions $\mathrm{F} : \chi \to \mathbb{R}$ as

$$\mathfrak{S}\mathrm{F}(\xi) = \lambda \sum_{x \in \mathbb{Z}} \left[\mathrm{F}(\xi^{x,x+1}) - \mathrm{F}(\xi) \right] + \gamma \sum_{x \in \mathbb{Z}} \left((-1)^{\xi_x} - 1 \right) \mathrm{F}(\xi).$$

Above $\xi^{x,y}$ is obtained from ξ by exchanging ξ_x and ξ_y.

From this result we deduce:

Corollary A.2. *For any $f = \sum_{\xi \in \chi} \mathrm{F}(\xi) \mathrm{H}_\xi \in \mathbf{L}^2(\mu_1)$, we have*

$$\mathcal{D}(\mu_1; f) = \langle f, -\mathcal{S} f \rangle_1 = \sum_{\xi \in \chi} \left\{ \frac{\lambda}{2} \sum_{x \in \mathbb{Z}} \left(\mathrm{F}(\xi^{x,x+1}) - \mathrm{F}(\xi) \right)^2 + \gamma \sum_{x \in \mathbb{Z}} \left((-1)^{\xi_x} - 1 \right) \mathrm{F}^2(\xi) \right\}$$

A.3 Quadratic functions

Recall Definition 2.2. In other words, we are mostly interested in *quadratic* functions f in $\mathbf{L}^2(\mu_1)$, which have zero average with respect to μ_1 and compact support. They correspond exactly to *degree 2 functions* as we already noticed in Remark A.1, but with the additional assumption that their support is compact.

The next propositions give some useful properties:

PROPOSITION A.3. *If $f \in \mathbf{L}^2(\mu_1)$ is of degree 2, then the following variational formula*

$$\sup_{g \in \mathbf{L}^2(\mu_1)} \left\{ 2\left\langle f, g \right\rangle_1 - \mathcal{D}(\mu_1; g) \right\}$$

can be restricted over degree 2 functions g.

Moreover, if the support of f is finite and included in Λ, then the supremum can be restricted to functions with support included in Λ.

Proof. This fact follows after decomposing g as $\sum_{\xi \in \chi} \mathrm{G}(\xi)\mathrm{H}_\xi$. Corollary A.2 and the orthogonality of Hermite polynomials imply that we can restrict the supremum over functions g of degree two (91). Moreover, if $x \neq y$, then $\langle (\omega_x^2 - 1)(\omega_y^2 - 1) \rangle_1 = 0$, and if x, y, z, t are all distinct, then $\langle \omega_x \omega_y \omega_z \omega_t \rangle_1 = 0$. This implies that the support of g can be restricted to the one of f, otherwise it would only increase the Dirichlet form. □

PROPOSITION A.4. *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of degree 2 functions in $\mathbf{L}^2(\mu_1)$. Suppose that $\{f_n\}$ weakly converges to f in $\mathbf{L}^2(\mu_1)$. Then, f is of degree 2.*

Moreover, if any f_n has support included in some finite subset Λ, then the support of f is included in Λ.

Proof. For all $n \in \mathbb{N}$, and $\xi \notin \chi_2$, the scalar product $\left\langle f_n, \mathrm{H}_\xi \right\rangle_1$ vanishes (by definition). From the weak convergence, we know that

$$\left\langle f_n, \mathrm{H}_\xi \right\rangle_1 \to \left\langle f, \mathrm{H}_\xi \right\rangle_1,$$

as n goes to infinity, for all $\xi \in \chi$. This implies: $\left\langle f, \mathrm{H}_\xi \right\rangle_1 = 0$ for all $\xi \notin \chi_2$. □

Note that the set denoted by $\mathcal{Q}$ and defined in Definition 2.2 contains cylinder quadratic functions in $\mathbf{L}^2(\mathbb{P}_1^\star)$. The conclusions of Propositions A.3 and A.4 can be restated for our purpose as:

COROLLARY A.5. *If $f \in \mathcal{Q}$, then the following variational formula*

$$\sup_{g \in \mathbf{L}^2(\mathbb{P}_1^\star)} \left\{ 2\mathbb{E}_1^\star[f, g] - \mathcal{D}(\mathbb{P}_1^\star; g) \right\}$$

can be restricted over functions g in $\mathcal{Q}$. Moreover, if $\{f_n\}_n$ is a sequence of functions in $\mathcal{Q}$ such that $\{f_n\}$ weakly converges to f in $\mathbf{L}^2(\mathbb{P}_1^\star)$, then f belongs to $\mathcal{Q}$.

B Proof of the weak sector condition

In this section we prove Proposition 5.7 that we recall here for the sake of clarity.

PROPOSITION B.1 (Weak Sector condition).
(i) There exists two constants $\mathrm{C}_0(\gamma, \lambda)$ and $\mathrm{C}_1(\gamma, \lambda)$ such that the following inequality hold for all $f, g \in \mathcal{Q}$:

$$\left| \ll \mathcal{A}^{\mathbf{m}} f, \mathcal{S} g \gg_\beta \right| \leqslant \mathrm{C}_0 \left\| \mathcal{S} f \right\|_\beta \left\| \mathcal{S} g \right\|_\beta.$$
$$\left| \ll \mathcal{A}^{\mathbf{m}} f, \mathcal{S} g \gg_\beta \right| \leqslant \mathrm{C}_1 \left\| \mathcal{S} f \right\|_\beta^2 + \frac{1}{2} \left\| \mathcal{S} g \right\|_\beta^2.$$

(ii) There exists a positive constant $\mathrm{C}(\beta)$ such that, for all $g \in \mathcal{Q}$,

$$\left\| \mathcal{A}^{\mathbf{m}} g \right\|_\beta \leqslant \mathrm{C}(\beta) \left\| \mathcal{S} g \right\|_\beta.$$

Proof. We prove (i). We assume that

$$g(\mathbf{m},\omega)=\sum_{x\in\mathbb{Z}}\psi_{x,0}(\mathbf{m})(\omega_{x+1}^2-\omega_x^2)+\sum_{\substack{x\in\mathbb{Z}\\k\geqslant 1}}\psi_{x,k}(\mathbf{m})\omega_x\omega_{x+k}$$

$$f(\mathbf{m},\omega)=\sum_{x\in\mathbb{Z}}\varphi_{x,0}(\mathbf{m})(\omega_{x+1}^2-\omega_x^2)+\sum_{\substack{x\in\mathbb{Z}\\k\geqslant 1}}\varphi_{x,k}(\mathbf{m})\omega_x\omega_{x+k}.$$

We denote by $\Delta^{\mathbf{m}}\psi$ the discrete Laplacian in the variable $\mathbf{m}$, that is

$$\Delta^{\mathbf{m}}\psi(\mathbf{m})=2\psi(\mathbf{m})-\psi(\tau_1\mathbf{m})-\psi(\tau_{-1}\mathbf{m}),$$

and $\tau_x\Delta^{\mathbf{m}}$ is the operator defined as

$$(\tau_x\Delta^{\mathbf{m}})\psi(\mathbf{m}):=\Delta^{\mathbf{m}}\psi(\tau_x\mathbf{m}).$$

Straightforward computations show that

$$\begin{aligned}\|\mathcal{S}g\|_\beta^2&=\frac{\gamma}{2}\mathbb{E}_\beta^\star\left[(\nabla_0\Gamma_g)^2\right]+\frac{\lambda}{2}\mathbb{E}_\beta^\star\left[(\nabla_{0,1}\Gamma_g)^2\right]\\&=\frac{4\gamma}{\beta^2}\sum_{\substack{x\in\mathbb{Z}\\k\geqslant 1}}\mathbb{E}[\psi_{x,k}^2]+\frac{2\lambda}{\beta^2}\sum_{x\in\mathbb{Z}}\mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}}\tau_x(\Delta^{\mathbf{m}}\psi_{x,0})\Big)^2\Big]\\&\quad+\frac{\lambda}{\beta^2}\sum_{k\geqslant 2}\mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}}\big[\tau_{-x}(\psi_{x,k})-\tau_{1-x}(\psi_{x,k})\big]\Big)^2\Big],\end{aligned}$$

$$\|\mathcal{S}f\|_\beta^2\geqslant\|\mathcal{S}^{\text{flip}}f\|_\beta^2=\frac{\gamma}{2}\mathbb{E}_\beta^\star\Big[\Big(2\sum_{\substack{z\in\mathbb{Z}\\k\geqslant 1}}\varphi_{z,k}(\mathbf{m})\omega_0\omega_k\Big)^2\Big]=\frac{2\gamma}{\beta^2}\sum_{k\geqslant 1}\mathbb{E}\Big[\Big(\sum_{z\in\mathbb{Z}}\varphi_{z,k}(\mathbf{m})\Big)^2\Big].\tag{92}$$

Now we deal with $\ll\mathcal{A}^{\mathbf{m}}g,\mathcal{S}f\gg_\beta$. From Proposition 5.1, and by definition,

$$\begin{aligned}\ll\mathcal{A}^{\mathbf{m}}g,\mathcal{S}f\gg_\beta&=-\sum_{z\in\mathbb{Z}}\mathbb{E}_\beta^\star\left[f,\tau_z(\mathcal{A}^{\mathbf{m}}g)\right]\\&=-\sum_{x,z\in\mathbb{Z}}\mathbb{E}\left[\varphi_{x,0}(\mathbf{m})\langle\omega_{x+1}^2-\omega_x^2,\tau_z(\mathcal{A}^{\mathbf{m}}g)\rangle_\beta\right]-\sum_{\substack{x,z\in\mathbb{Z}\\k\geqslant 1}}\mathbb{E}\left[\varphi_{x,k}(\mathbf{m})\langle\omega_x\omega_{x+k},\tau_z(\mathcal{A}^{\mathbf{m}}g)\rangle_\beta\right]\\&=\frac{2}{\beta^2}\sum_{x\in\mathbb{Z}}\mathbb{E}\Big[\frac{\tau_x(\Delta^{\mathbf{m}}\psi_{x,0})}{\sqrt{m_xm_{x+1}}}\sum_{z\in\mathbb{Z}}\tau_{-z}(\varphi_{z,1})\Big]\\&\quad+\frac{1}{\beta^2}\sum_{x\in\mathbb{Z}}\mathbb{E}\Big[\Big(\frac{\tau_1\psi_{x,1}}{\sqrt{m_xm_{x+1}}}-\frac{\psi_{x,1}}{\sqrt{m_{x+1}m_{x+2}}}\Big)\sum_{z\in\mathbb{Z}}\tau_{-z}(\varphi_{z,2})\Big]\\&\quad+\frac{1}{\beta^2}\sum_{k\geqslant 2}\sum_{x\in\mathbb{Z}}\mathbb{E}\Big[\Big(\frac{\tau_1\psi_{x,k}}{\sqrt{m_xm_{x+1}}}-\frac{\psi_{x,k}}{\sqrt{m_{x+k}m_{x+k+1}}}\Big)\sum_{z\in\mathbb{Z}}\tau_{-z}(\varphi_{z,k+1})\Big]\\&\quad+\frac{1}{\beta^2}\sum_{k\geqslant 2}\sum_{x\in\mathbb{Z}}\mathbb{E}\Big[\Big(\frac{\tau_{-1}\psi_{x,k}}{\sqrt{m_xm_{x+1}}}-\frac{\psi_{x,k}}{\sqrt{m_{x+k}m_{x+k-1}}}\Big)\sum_{z\in\mathbb{Z}}\tau_{-z}(\varphi_{z,k-1})\Big].\end{aligned}$$

From Cauchy-Schwarz inequality, and recalling $1/\sqrt{m_0 m_1} \leqslant C$ ($\mathbb{P}$-a.s.), we obtain the following bound:

$$\| \ll \mathcal{A}^{\mathbf{m}} g, \mathcal{S} f \gg_\beta \| \leqslant \frac{2C}{\beta^2} \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}} \tau_x (\Delta^{\mathbf{m}} \psi_{x,0}) \Big)^2 \Big]^{1/2} \mathbb{E}\Big[\Big(\sum_{z\in\mathbb{Z}} \tau_{-z} \varphi_{z,1} \Big)^2 \Big]^{1/2} \tag{93}$$

$$+ \frac{3C}{\beta^2} \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}} \tau_1 \psi_{x,1} - \psi_{x,1} \Big)^2 \Big]^{1/2} \mathbb{E}\Big[\Big(\sum_{z\in\mathbb{Z}} \tau_{-z} \varphi_{z,2} \Big)^2 \Big]^{1/2} \tag{94}$$

$$+ \frac{3C}{\beta^2} \sum_{k\geqslant 2} \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}} \tau_1 \psi_{x,k} - \psi_{x,k} \Big)^2 \Big]^{1/2} \mathbb{E}\Big[\Big(\sum_{z\in\mathbb{Z}} \tau_{-z} \varphi_{z,k+1} \Big)^2 \Big]^{1/2} \tag{95}$$

$$+ \frac{3C}{\beta^2} \sum_{k\geqslant 2} \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}} \tau_{-1} \psi_{x,k} - \psi_{x,k} \Big)^2 \Big]^{1/2} \mathbb{E}\Big[\Big(\sum_{z\in\mathbb{Z}} \tau_{-z} \varphi_{z,k-1} \Big)^2 \Big]^{1/2}. \tag{96}$$

Now we are going to use twice the trivial inequality $\sqrt{ab} \leqslant a/\varepsilon + \varepsilon b/2$ for a particular choice of $\varepsilon > 0$: in (93) we take $\varepsilon = \gamma/C$ and in (94) we take $\varepsilon = 2\gamma/(3C)$. This trick gives the final bound

$$\| \ll \mathcal{A}^{\mathbf{m}} g, \mathcal{S} f \gg_\beta \| \leqslant \frac{2C^2}{\gamma\beta^2} \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}} \tau_x (\Delta^{\mathbf{m}} \psi_{x,0}) \Big)^2 \Big] + \frac{2\gamma}{\beta^2} \sum_{k\geqslant 1} \mathbb{E}\Big[\Big(\sum_{z\in\mathbb{Z}} \varphi_{z,k}(\tau_{-z}\mathbf{m}) \Big)^2 \Big]$$
$$+ \frac{9C^2}{\gamma\beta^2} \sum_{k\geqslant 2} \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}} \tau_1 \psi_{x,k} - \psi_{x,k} \Big)^2 \Big].$$

Recalling (92), we obtain

$$\| \ll \mathcal{A}^{\mathbf{m}} g, \mathcal{S} f \gg_\beta \| \leqslant \frac{9C^2}{\gamma\lambda} \vert\vert\vert \mathcal{S} g \vert\vert\vert_\beta^2 + \frac{1}{2} \vert\vert\vert \mathcal{S} f \vert\vert\vert_\beta^2.$$

If we use the Cauchy-Schwarz inequality, we get:

$$\ll \mathcal{A}^{\mathbf{m}} g, \mathcal{S} f \gg_\beta^2 \leqslant \frac{18C^2}{\gamma\lambda} \vert\vert\vert \mathcal{S} g \vert\vert\vert_\beta^2 \; \vert\vert\vert \mathcal{S} f \vert\vert\vert_\beta^2.$$

We have proved (i) with $C_0 = \sqrt{18C^2/(\gamma\lambda)}$ and $C_1 = 9C^2/(\gamma\lambda)$. Now we turn to (ii). From Lemma 5.5 and Statement (i),

$$\ll \mathcal{A}^{\mathbf{m}} g, j^{S}_{0,1} \gg_\beta = \ll \mathcal{S} g, j^{A}_{0,1} \gg_\beta \leqslant \vert\vert\vert \mathcal{S} g \vert\vert\vert_\beta \vert\vert\vert j^{A}_{0,1} \vert\vert\vert_\beta.$$

Moreover, from Statement (i), we also get, for all $f \in \mathcal{Q}_0$,

$$-2 \ll \mathcal{A}^{\mathbf{m}} g, \mathcal{S} f \gg_\beta \leqslant \vert\vert\vert \mathcal{S} f \vert\vert\vert_\beta^2 + \frac{2C}{\gamma\lambda} \vert\vert\vert \mathcal{S} g \vert\vert\vert_\beta^2.$$

As a result, the variational formula (44) for $\vert\vert\vert \mathcal{A}^{\mathbf{m}} g \vert\vert\vert_\beta^2$ gives:

$$\vert\vert\vert \mathcal{A}^{\mathbf{m}} g \vert\vert\vert_\beta^2 \leqslant \frac{1}{\lambda\chi(\beta)} \ll \mathcal{A}^{\mathbf{m}} g, j^{S}_{0,1} \gg_\beta^2 + \frac{9C^2}{\gamma\lambda} \vert\vert\vert \mathcal{S} g \vert\vert\vert_\beta^2 \leqslant \left(\frac{\vert\vert\vert j^{A}_{0,1} \vert\vert\vert_\beta^2}{\lambda\chi(\beta)} + \frac{9C^2}{\gamma\lambda} \right) \vert\vert\vert \mathcal{S} g \vert\vert\vert_\beta^2.$$

The result is proved. □

C Tightness

In this section we prove the tightness of the sequence $\{\mathfrak{Y}^N\}_{N\geqslant 1}$, by using standard arguments. First, let us recall that the space $\mathfrak{H}_{-k}$ is equipped with the norm defined as

$$\|\mathcal{Y}\|_{-k}^2 = \sum_{n\geqslant 1} (\pi n)^{-2k} \big|\mathcal{Y}(\mathbf{e}_n)\big|^2.$$

Theorem C.1. *The sequence $\{\mathfrak{Y}^N\}_{N\geqslant 1}$ is tight in $\mathcal{C}([0,T],\mathfrak{H}_{-k})$.*

Proof. The tightness of the sequence $\{\mathfrak{Y}^N\}$ follows from two conditions (see [16], page 299):

$$\lim_{A\to\infty}\limsup_{N\to\infty} \mathbb{P}^\star_{\mu_\beta^N}\Big[\sup_{0\leqslant t\leqslant T}\|\mathcal{Y}^N_{t,\mathbf{m}}\|_{-k} > A\Big] = 0 \tag{97}$$

(98)

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{P}^\star_{\mu_\beta^N}\Big[w(\mathcal{Y}^N_{\mathbf{m}},\delta) > \varepsilon\Big] = 0, \qquad \text{for all } \varepsilon > 0, \tag{99}$$

where the modulus of continuity $w(\mathcal{Y},\delta)$ is defined by

$$w(\mathcal{Y},\delta) = \sup_{\substack{\|t-s\|<\delta\\ 0\leqslant s\leqslant t\leqslant T}} \|\mathcal{Y}_t - \mathcal{Y}_s\|_{-k}.$$

Let us remind the decomposition of $\mathcal{Y}^N_{t,\mathbf{m}}$ given in (20):

$$\mathcal{Y}^N_{t,\mathbf{m}}(H) = \mathcal{Y}^N_{0,\mathbf{m}}(H) + \int_0^t D\mathcal{Y}_{s,\mathbf{m}}(\Delta_N H)ds + \mathfrak{M}^{1,N}_{t,\mathbf{m},f_k}(H) + Z^N_{t,\mathbf{m},f_k}(H),$$

where $\mathfrak{M}^{1,N}_{t,\mathbf{m},f_k}(H)$ is the martingale defined in Subsection 3.3, and $Z^N_{t,\mathbf{m},f_k}(H)$ is defined as the sum of the remaining terms in the decomposition. On the first hand,

$$\mathbb{E}^\star_{\mu_\beta^N}\left[\sup_{0\leqslant t\leqslant T}\Big(Z^N_{t,\mathbf{m},f_k}(H)\Big)^2\right]$$

can be estimated by the proof of Lemma 3.2 and Theorem 3.3. On the other hand,

$$\mathbb{E}^\star_{\mu_\beta^N}\left[\Big(\mathfrak{M}^{1,N}_{t,\mathbf{m},f_k}(H)\Big)^2\right]$$

can be computed explicitly. □

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